A SYNOPSIS OF ALGEBRA.

Being the Poſthumous Work of

JOHN ALEXANDER,

Of *BERN* in *Swiſſerland.*

To which is added an

APPENDIX

By

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To the Right Worſhipful
Sir *Francis Child*, Knt.
PRESIDENT;
The Worſhipful
Thomas Lockington, Eſq;
TREASURER;
And the reſt of the Worſhipful the
GOVERNOURS
OF
Chriſt's-Hoſpital, *London*,
(At whoſe Charge this BOOK is Printed)
And alſo
The Worthy Mr. *Edward Brewſter*,
(The generous Donor of the MANUSCRIPT)
This *SYNOPSIS*,
For the uſe of the two *Mathematical Schools*,
Corrected, Augmented and Reviſed,
As a Mark of his Gratitude, is
Humbly Inſcrib'd by
The *Tranſlator*.

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The

The Reader is deſired to mend with his Pen theſe following ERRATA, among others of leſs Note.

PAG. 3. lin. 2. read *Literal.* P. 13. line laſt r. *Unity*. P. 93. and the following Pages r. *Plane* for *Plain*. P. 103. l. 19. r. *extract*. P. 151. for *hyp.* r. *hypotheſis.* In the Appendix, P. 9. l. 15. for *Quo.* r. *Quotient*. P. 59. l. 1. r. *conſtructing*. P. 107. l. 5. dele *it*.

AN

AN INTRODUCTION TO *Literal ALGEBRA.*

LITERAL ALGEBRA is the Science of Quantities, the end of which is the Solution of Problems, and the Diſcovery of Theorems, contain'd in thoſe Problems. It is call'd *Literal*, to diſtinguiſh it from *Numerical.* It goes alſo by the Name of *The Analytick Art*, and *Univerſal Mathematicks.*

Quantities (whether they be Numbers or Magnitudes) are diſtinguiſh'd into *Given* and *Sought.*

They are both uſually denoted by certain Letters of the Alphabet, tho' different Authors have a different Method.

We ſhall expreſs the *Given* Quantities by the firſt Letters of the Alphabet, *a*, *b*, *c*, *d*, *&c.* and the *Sought* by the laſt, *x*, *y*, *z*. We uſe the little Letters, becauſe they are made with greater eaſe, and prevent confuſion; whereas the

Capital Letters are commonly apply'd to denote the Lines of Geometrical Figures.

The Quantities denoted by the ſame Letters (for Example, a and a; b and b) in the ſame Queſtion, are underſtood to be of the ſame Value and Subject; that is, they denote the ſame or equal Lines or Numbers, and therefore, by prefixing Numbers they may be joyn'd into one; as $2a$, $3b$, $4c$, $\frac{1}{2}a$, $\frac{2}{3}b$, $\frac{3}{4}c$, *&c.*

But thoſe which are denoted with different Letters, are of different Value; and are not joyn'd together, except by Multiplication, or by the means of the Signs $+$ or $-$, the firſt ſignifies *more*, the ſecond *leſs*; one is a Note of Addition, and the other of Subtraction.

And it is to be obſerv'd, that theſe Marks $+$ and $-$, are always ſet before their Quantities, and are thereſore referr'd to the following Quantity; and the Quantities which have no Mark prefix'd to them (as Initials) are ſuppos'd to have $+$.

Notwithſtanding there may be an equality between Quantities of a different value; for Example, as $2b$ may be equal to $3a$ [as 5 *Dutch* Florins are equal to 9 *Engliſh* Shillings].

For denoting Equality (which is the chief Buſineſs of *Algebra*) theſe Marks are commonly us'd, $=$ or $\propto$; as $2b=3a$; that is, $2b$ equal to $3a$, *&c.*

Further; the Inſtruments which *Algebra* makes uſe of for the Solution of Queſtions and Diſcovery of Truths, are Logiſtical Operations, and particularly Equations.

One part of *Algebra* is *Numerical*, which proceeds by Numbers; the other *Literal* which uſes Univerſal Symbols, ſuch as the Letters of the Alphabet; by which, Numbers, as well as Magnitudes (as was ſaid before) are ſignified.

Hence *Algebra* it ſelf is divided into *Numerical* and *Literal*, the former of which is more Ancient, and the other more Modern; and, as far as we can find, *Francis Vieta* was the firſt Author of it; ſince whoſe time it has not only found very great Improvements, but has, in a manner, been brought to the very height of Perfection, by the Labours of ſeveral eminent and extraordinary Perſons.

Now the Difference betwixt the two *Algebras* conſiſts chiefly in this, that in every Problem, the Literal does not only find out the very thing ſought for, as the *Numerical* does; but, at the ſame time diſcovers a general Rule and Mode of Conſtruction, which the *Numerical* cannot do, becauſe all the Footſteps of the Proceſs are confounded and loſt in it; which the *Literal* always keeps intire, that they may readily be made uſe of backward again in order to Compoſition.

Literal Algebra, as well as *Numerical*, comes to be conſider'd in Quantities. 1. Abſolute, and thoſe Integers. 2. Broken, or Fractions. 3. Surd or Irrational. 4. Binomials.

But *Numerical Algebra*, in this place is partly preſuppos'd, and partly will be eaſily known by the *Literal*, both uſing the ſame Rules.

Here follows the *Algebra* of Intire Quantities.

ADDITION.

Addition joins together all propos'd Quantities, keeping the ſame Signs $+$ and $-$.

Examples.

1. *In ſimple Quantities of the ſame kind.*

	$2a$		a		$3d$
add.	a	add.	$-a$		$5d$
the Sum.	$2a+a$	the Sum.	$a-a$	Sum.	$8d$
that is	$3a$	that is	0 *		

* For quantities of the ſame name being affected by contrary Signs, in Addition deſtroy one another.

	$2c$		$5b$		$2b$
add.	$4c$	add.	$-2b$	add.	$-5b$
	$9c$	Sum.	$5b-2b$	Sum.	$2b-5b$
Sum.	$15c$	that is	$3b$	that is	$-3b$

2. *In ſimple Quantities of a different kind.*

	a		$4c$		$3c$
add.	b	add.	$2d$	add.	$2e$
Sum.	$a+b$	Sum.	$4c+2d$	and	f
				Sum.	$3c+2e+f$

	b		$3f$		$2d$
add.	$-a$	add.	1	add.	-2
Sum.	$b-a$	Sum.	$3f+1$	Sum.	$2d-2$

3. *In compound Quantities of the ſame kind.*

	$2a + 3c + 4d$		$a + b - 2c$
add.	$5d + a + 2c$	add.	$a - b + 3c$
Sum.	$3a + 5c + 9d$	Sum.	$2a + c$

Here we muſt take notice that the quantities to be added are ſometimes confuſedly ſet down, which notwithſtanding may be eaſily ſet in good order, ſo that the ſame ſorts of quantities may anſwer one another.

The ſame likewiſe is uſual in Subtraction and Diviſion, where if the Signs are different, the Addition will be made by Subtraction, ſetting the Sign of the Greater before.

	$6a + 2b - d$		$4f + 2b - 5c$
add.	$-4d - 2a + 3b$	add.	$3c - f - 2b$
Sum.	$4a + 5b - 5d$	Sum.	$3f - 2c$

	$2aa + 2b - 5c + 9d$		$bb + 2abc - ab + ac$
add.	$-9d + aa - 6b + 5c$	add.	$bb - abc + ab - 3ac$
Sum.	$3aa - 4b$	Sum.	$2bb + abc - 2ac$

	$5d - 3ff + 2$		$3ff + 2d - 2$
add.	$-2d + 4ff - 6$	add.	$-4ff - 5d + 6$
Sum.	$3d + ff - 4$	Sum.	$-ff - 3d + 4$

4. *In compound Quantities of a different kind.*

	$3a + c$		$5d - 4c + 1$
add.	$2b + d - f$	add.	$-a - 2b + 45$
Sum.	$3a + c + 2b + d - f$	Sum.	$5d - 4c - a - 2b + 46$

SUBSTRACTION.

SUbtraction joins together two propos'd Quantities, changing all the Signs of the Quantity to be ſubducted.

EXAMPLES.

1. *In ſimple Quantities of the ſame kind.*

from	$5a$	from	$2bc$	from	$4acd$
ſubt	$2a$	ſubt.	bc	ſubt.	$4acd$
rem.	$5a-2a$	rem.	$2bc-bc$	rem.	$4acd-4acd$
that is	$3a$	that is	bc	that is	0

from	$-df$	from	$4b$	from	$3c$
ſubt.	$-df$	ſubt.	$6b$	ſubt.	$-3c$
rem.	$-df+df$	rem.	$4b-6b$	rem.	$3c+3c$
that is	0	that is	$-2b$	that is	$6c$

2. *In ſimple Quantities of a different kind.*

from	a	from	$3d$	from	c
ſubt.	b	ſubt.	$2c$	ſubt.	a, b, and d
rem.	$a-b$	rem.	$3d-2c$	rem.	$c-a-b-d$

from	a and b	from	df	from	$4bc$
ſubt.	c, d, and e	ſubt.	$2bd$	ſubt.	4
re.	$a+b-c-d-e$	rem.	$df+2bd$	rem.	$4bc-4$

Note, If it is doubtful which of the Quantities is Greater or Leſſer, the difference is generally expreſs'd by ſome other Sign.

as $b \backsim c$ that is $b-c$ or $c-b$

3. *In compound Quantities of the ſame kind.*

* Examples where the Signs are the ſame.

from	$4a + 3b$	from	$6c - 5b$
ſubt.	$2a + 2b$	ſubt.	$4c - 3b$
rem.	$2a + b$	rem.	$2c - 2b$

from	$3c + 2a - b$	from	$5d + 12$
ſubt,	$c + 3a - 2b$	ſubt.	$6d - 5$
rem.	$2c - a + b$	rem.	$-d + 17$

from	$ab - 4cd$	from	$3c + 2a - d$
ſubt.	$2ab - 5cd$	ſubt.	$2c + 3a$
rem.	$-ab + cd$	rem.	$c - a - d$

from	$4a-2c$	let $a=9$	$\because$	$4a - 2c = 32$
ſubt.	$2a-6c$	$c=2$	and	$2a - 6c = 6$
rem.	$2a+4c$		and	$2a + 4c = 26$

from	$5d+2b-c$	let $d=12$	$\because$	$5d + 2b = 66$
ſubt.	$4d+5b-c$	$b=3$		$4d + 5b = 63$
rem.	$d-3b$			$d - 3b = 3$

from	$6a + 9b - 3c + 4d$
ſubt.	$7a + 2b - 3c + 8d$
rem.	$-a + 7b \qquad - 4d$

---

* Where if the Signs are the ſame; the remainder will alſo be the ſame; except where the quantity before is greater, than by Subtracting, on the contrary, the remainder will have the contrary Sign. But if the Signs are different, the Subtraction will be made by Addition, prefixing the Sign of the quantity above.

Examples where the Signs are different.

from	$4a+3b$	Let $a=3$	$\therefore 4a+3b=18$
ſubt.	$2a-2b$	$b=2$	$2a-2b=2$
rem.	$2a+5b$		$2a-5b=16$

from	$2b-2c+d$	from	$5bc-bb$
ſubt.	$b+4c-d$	ſubt.	$3bb-3bc$
rem.	$b-6c+2d$	rem.	$8bc-4bb$

from	$aa-4d+6c-2a$
ſubt.	$4a+2aa-5c-6d$
rem.	$2d-aa+11c-6a$

from	$6bb+3aa-3c+9b$
ſubt.	$4bb-4aa-6e-5b$
rem.	$2bb+7aa+3c+14b$

from	$ccc+bbd-aac-eee$
ſubt.	$2bbd-2ccc+eee$
rem.	$3ccc+bbd-aac-2eee$

from	$7ab+dd-10$
ſubt.	$9ab-2dd+18$
rem.	$-2ab+3dd-28$

4. *In*

4. *In Compound Quantities of a different kind.*

from $aa-bb$	from $bc-dd$
ſubt. $aa-ab+ac$	ſubt. $ad+bd-2cd$
rem. $ab-bb-ac$	rem. $bc-dd-ad-bd+2cd$

from $5b-2a$ ſubtr. $3c+d-4f$

rem. $5b-2a-3c-d+4f$

let $b=10$

$a=5$

$c=6$

$d=7$

$f=4$

$\because$ $5b-2a=40$

$3c+d-4f=9$

$5b-2a-3c-d+4f=31$

for $5b+4f=66$

ſub. $2a+3c+d=35$

rem. 31

MULTI-

MULTIPLICATION

JOyns together the Letters of both Factors, without any thing between; or places the Multiplier by the Multiplicand. But the Numbers ſet before the Quantities are multiply'd after the common way. The ſame Signs (whether + or —) give in the Product +, but different Signs give —.

EXAMPLES.

1. *In Simple Quantities.*

Multiply a
by b
the Product will be ab

Multiply a
by a
product. aa or a^2
that is a Square
or a Second Power

Mult. aa or a^2
by a
Prod. aaa or a^3
that is a Cubick
or a Third Power

mult. d by b by a
$\underline{}$
db
$\underline{a}$
dba prod.

mult. c by 12
prod. $12c$

mult. $4a$
by $\underline{2b}$
prod. $8ab$

mult. $3bd$
by $\underline{2ab}$
$6bbad$

mult. c^3 by $5d^2$
prod. $5d^2c^3$

mult. $\underline{5cd \text{ by } 3ab}$
prod. $15abcd$

mult. $2ac$ by 10
prod. $20ac$

mult. $6b^4$ by $3b^2$
prod. $18b^6$

mult. $4ab^2$ by $4ca^2$
prod. $16a^3b^2c$

mult. $9aacdf$
by $\underline{5accf}$
prod. $45a^3c^3f^2d$

Where the Multiplier and Multiplicand conſiſt of ſeveral Members, all the particular Products are to be ſum'd up together, in order to obtain the total Product.

mult. $a+b$ by c
prod. $ac+bc$

mult. $b-a$ by $2d$
prod. $2bd-2ad$

---

mult. $a+2b-3c+d$ by $3e$
prod. $3ae+6be-9ce+3de$
mult. $6b-2a-c-4f$ by $2ab$
prod. $12ab^2-4a^2b-2abc-8abf$
mult. $c+b$ by $a+d$
prod. $ac+ab+cd+bd$

mult. $b-a$ by $c-d$
prod. $bc-ac-bd+ad$
mult. $a+b$ by $a+b$
prod. $a^2+2ab+b^2$
mult. $a-b$ by $a-b$
prod. $a^2-2ab+b^2$
mult. $b+c$ by $b-c$
prod. b^2-c^2
mult. $b+c-2d$ by $b+c-2d$
prod. $b^2+2bc-4bd+c^2-4cd+4d^2$
mult. $2a+4b$ by $3c-5d$
prod. $6ac+12bc-10ad-20bd$
mult. $6f-d-2$ by $f+2d$
prod. $6f^2+11df-2d^2-2f-4d$
mult. $3b^2-2a^2+5$ by $9a+4b-1$
prod. $27ab^2-18a^3+45a+12b^3$, *&c.*

Farther it may be noted, that Multiplication is ſometimes expreſs'd after this manner;

$a+b-c \times d+e$
or $\underline{a+b-c \quad d+e}$
or thus $a+b-c : d+e$

DIVISION

REſolves the Dividend into the Diviſor and Quotient, which being Multiply'd into the whole Diviſor, reſtores the Dividend again. But if Numbers are prefix'd, the Diviſion is performed in the common way, and the ſame Signs make the Quotient $+$, different Signs make it $-$, as in Multiplication.

Or Diviſion, as it were, extracts the Diviſor from the Dividend, and takes the Remainder for the Quotient.

And this is ſufficient when the Diviſor is a Simple Quantity; but when it is a Compound one, then we proceed to find out the Quotient in the firſt Letter or Number of the Diviſor, juſt as it is in the common Practice of Numbers.

EXAMPLES.

1. *In Simple Quantities.*

divid. aa by a | a quotient   divid. bc by b | c quotient.
divid. $2abc$ by $2b$ | ac quotient
divid. $9cd$ by $3c$ | quotient $3d$
divid. $3f$ by f | quotient 3
divid. $4a^2b^2$ by $2b^2$ | quotient $2a^2$
divid. $6c^3b^4$ by $2c^2b^3$ | quotient $3bc$
divid. $3a^2b^3$ by $3ab^2$ | quotient ab
divid. $18cd$ by 9 | quotient $2cd$
divid. $4a^3b^2c$ by $5a^2b$ | quotient $\frac{4}{5}abc$ or $\frac{4abc}{9}$
divid. $6b^2c^3d^4$ by $4bc^2d^3$ | quotient $1\frac{1}{2}bcd$
divid. $12a^4c^4d^3$ by $4a^2c$ | quotient $3a^2c^3d^3$
divid. $5df$ by $5df$ | the quotient will be 1. For if the Diviſor be equal to the Dividend the Quotient will always be Unity.

2. *In*

2. *In Compound Quantities.*

divid. $ab+bc$ by b | quotient will be $a+c$
divid. $bc-cd$ by c | quotient $b-d$
divid. $af+ag+df+dg$ by $f+g$ | quotient $a+d$
divid. $bd-bc-ad+ac$ by $d-c$ | quotient $b-a$
div. $6a^2b+12abc-10a^2c^4-20ac^5$ by $2a^2+4ac$ | qu. $3b-5c^4$
divid. d^2+d by $d+1$ | quotient d
divid. $2f^3-f^2$ by $2f-1$ | quotient f^2.
divid. a^2-b^2 by $a+b$ | quotient $a-b$
divid. a^2-b^2 by $a-b$ | quotient $a+b$
divid. $b^3-3b^2a+3ba^2-a^3$ | quotient $b^2-2ab+a^2$
by $b-a$
ſubtr. b^3-b^2a
remain $-2b^2a+3ba^2-a^3$ | $-2ba$
$b-a$
ſubtr. $-2b^2a+2ba^2$
remains $+ba^2-a^3$ | $+a^2$
$b-a$
ſubtr. $+ba^2-a^3$
div. $8ab^3-12b^2cd+16b^6+10abc^3-15c^4d+20b^4c^3$
by $4b^2+5c^3$

$$\begin{array}{ll|l}
\text{1. member]} & 8ab^3+10abc^3 & 2ab \\
\text{2. member]} & -12b^2cd-15c^4d & -3cd \\
\text{3. member]} & +16b^6+20b^4c^3 & +4b^4
\end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}\text{quotient.}$$

[Proof by Multiplication.]

divid. c^3+d^3 by $c+d$ | quotient c^2+d^2-cd
divid. $ab+3b-2a-6$ by $b-2$ | quotient $a-3$

Further, If the Diviſion cannot be made after the following manner, then, by placing the Diviſor under the Dividend, there will be a Fraction (yet ſo as to include the Quotient) as is plain from what follows.

$$\text{divid.}\left\{\begin{array}{l} ac \text{ by } b \\ ef \text{ by } bc \\ c \text{ by } d \\ def \text{ by } gb \end{array}\right\}\text{the Quotient will be}\left\{\begin{array}{l} \frac{ac}{b} \\ \frac{ef}{bc} \\ \frac{c}{d} \\ \frac{def}{gb} \end{array}\right.$$

$$\text{Thus } a+b \text{ by } d, \text{ will be } \frac{a+b}{d}$$

$$b^2+c^2 \text{ by } e+f \qquad \frac{b^2+c^2}{e+f}$$

$$\frac{2a\times 3c-d}{b-c-d} \text{ by } b+c+d$$

$$dbc+af \text{ by } bc \mid d+\frac{af}{bc}$$

$$bde+3c \text{ by } d \mid be+\frac{3c}{d}$$

$$efg+fg \text{ by } ef \mid g+\frac{fg}{ef} \text{ that is } \frac{g}{f}$$

In like manner may any Diviſion be expreſs'd by Symbols, tho', in fact, it cannot be perform'd.

$$\frac{a^3-b^3}{a+b}$$

The

The Rule of PROPORTION

FROM two or three given Terms, rightly diſpos'd, by the help of Multiplication and Diviſion, as in the Practiſe of Numbers, produces the third or fourth Proportional ſought.

And the Diſpoſition of Quantities in a proportional Series, is uſually denoted after this manner;

$$\frac{(1)}{a.}\quad\frac{(2)}{b::}\quad\frac{(2)}{b.}\quad\frac{(3)}{X}$$

that is, as a to b: ſo b to X

$$\begin{array}{cccc}(1) & (2) & (3) & (4)\\ a. & b:: & c. & X\end{array}$$

as a to b: ſo c to X

Or thus;

$$a \div b : \quad b \div X$$
$$a \div b : \quad c \div X$$

Now as to the Practiſe it ſelf, we muſt obſerve,

1. That if the ſame Letters occur in the firſt or ſecond, or in the the firſt and third Term, they are thrown out before the Operation by the means of Diviſion; which muſt alſo be underſtood of the prefix'd Numbers.

2. But if the firſt Term be different from the reſt [at leaſt in part] as it commonly happens, then the fourth ſought always comes out in the form of a Fraction.

Ex-

Examples in Simple Quantities.

Terms given.

1.) $a. \quad \underbrace{2a :: 2a}_{\displaystyle \frac{4\cancel{a}a}{\cancel{a}}} \;\Big|\;$ $4a$ the Third Sought, which being preſuppos'd $=$ to x, then x will be $=$ to $4a$.

Or more compendiouſly thus.

$\cancel{a}. \quad \underbrace{2\cancel{a} :: 2a}_{4\cancel{a}a}$ the third ſought.

2.) $b. \quad \underbrace{ab :: ab}_{\displaystyle \frac{aabb}{b}} \;\Big|\;$ aab the third ſought.

or thus, $\cancel{b}. \quad \underbrace{a\cancel{b} :: ab}_{} \;\Big|\; aab$

3.) $2c. \quad \underbrace{d :: d}_{} \;\Big|\; \frac{dd}{2c}$ the third ſought.

4.) $\cancel{3}a. \quad \underbrace{b :: \cancel{3}b}_{} \;\Big|\; \frac{bb}{a}$ the third ſought.

5.) $\cancel{a}. \quad \underbrace{c :: 2\cancel{a}b}_{2bc}$ the fourth ſought.

6.) $2\cancel{b}. \quad \underbrace{3a :: \overset{}{\underset{\cancel{2}}{\cancel{4}\cancel{b}d}}}_{6ad}$ the fourth ſort.

7.) $d. \quad \underbrace{5b :: a}_{} \;\Big|\; \frac{5ab}{d}$ the fourth ſought.

C

8.)

8. $x\varnothing a^2 \quad \underbrace{7c :: \quad x\varnothing d^2} \;\Big|\; \frac{7cd^2}{a}$ the fourth ſought.

Examples in Compound Quantities.

9. $a. \quad \underbrace{a+b :: \quad a+b}_{aa+2ab+bb}$

div. a $\;\Big|\; a+2b+\frac{bb}{a}$ the fourth ſought.

10.) $a. \quad \underbrace{a+b :: \quad a-b}_{aa-bb}$

div. a $\;\Big|\; a-\frac{bb}{a}$ the fourth ſought.

11.) $b. \quad \underbrace{b+c :: \quad 2b-c}_{2bb+bc-cc}$

div. b $\;\Big|\; 2b+c-\frac{cc}{b}$ the fourth ſought.

12.) $a+2b \quad \underbrace{a :: \quad 2a-b}$

$\Big|\; \frac{2aa-ab}{a+2b}$ the fourth ſought.

13.) $3b-3a \quad \underbrace{2a+c :: \quad 3d}$

$\frac{2ad+cd}{b-a}$ the fourth ſought.

14.) $2d+c \quad \underbrace{2b+d :: \quad 2d-c}$

$\frac{4bd+2dd-2bc-cd}{2d+c}$ the fourth ſought.

The GENESIS and ANALYSIS
OF
POWERS

[*By others call'd Involution and Evolution; uſing theſe Marks,* & *ɯ.*]

POWERS are nothing elſe but Products ariſing from a continual Multiplication of any aſſum'd Side or Root.

For ſuppoſing that any Root is the firſt Power of it ſelf; or that the firſt Power ariſes, when the Root is multiply'd into Unity.

If then the ſame Root be multiply'd

into	it ſelf
	its Square
	its Cube
	its Biquadrate or Square-Square
	its Square-Cube

it produces	its Square	which is the	Second	Power
	its Cube		Third	
	its Biq. or SquareSq.		Fourth	
	its Square-Cube		Fifth	
	its Cube-Cube		Sixth	

whoſe Indices or Numeral Exponents are	2	3 or any other ariſing from them	but the Literal are	Q
	3			C
	4			Qq
	5			Qc
	6			Cc

And ſo on, *ad infinitum*, according to the number of their Dimenſions, of which they are compounded.

But the more compounded the Powers are, the higher they are eſteem'd to be; and therefore they are not diſtinguiſh'd by bare Name or Denomination only, but by their Order and Degrees, which conſtitute a ſort of progreſſional Scale, in which the Quantities do, as it were, aſcend and deſcend.

But we muſt alſo obſerve, that the Powers, which are above the Cubick, are not only produced from the Root it ſelf, multiply'd into the Degree next going before, but alſo from the mutual Multiplication of the inferiour Powers, as their very Names declare.

For Example, The Bi-quadrate (or Square-Square) ariſes not only from the Root multiplied into the Cube, but alſo from the Square multiply'd into it ſelf; whence this Power (*i. e.* the Fourth) is alſo a Square. And the like you may judge of the reſt.

Further: Every Root is either (1.) Simple, that is, of one Name, as a, or $2a$, or $3a$, *&c.* for the Numbers ſet before the Letters do not increaſe the Name; or (2.) compounded of more Names, that is, binomials, trinomials, *&c.* as, $a+b$, or $a-b$, $a+b+c$, $a+b-c$, $a-c+b$, $a-b-c$, *&c.*

Moreover, every Root is either rational, as a, $2a$, $\frac{1}{2}a$, *&c.* or irrational, as $\sqrt{a}$, $\sqrt{ab}$; and as the Roots are, ſuch alſo are their Powers; but in this place we ſhall treat only of thoſe that are rational.

The *Geneſis* of any Power is the Production of the ſame Power by Multiplication from a given Root.

The *Analyſis* of any Power is the Extraction of the Root from the given Power [by the means of a certain ſort of Diviſion.]

Wherefore, in the *Geneſis* the Root is given, and the Power ſought; but in the *Analyſis* the Power is given, and the Root ſought.

But then the Root of any Power (tho' a rational one) is not always a *Side*, properly ſo call'd (or a Firſt Power) but oftentimes the very Powers themſelves do perform the uſe and office of a Root. So aa, or a^2 (the Second Power) may ſerve for a Root, from whence we may make any Power whatſoever.

Therefore the *Geneſis* of Powers is eaſie from a Simple Root: That is to ſay, (1.) If the Root be a Side (or Firſt Power) Let there be annex'd to it a Simple Number, ſuch as agrees to the Nature of that Power, and plac'd above to the Right Hand, as the Index of the proper Multiplication: As if the Root be a, its Square will be a^2, Cube a^3, *&c.* if the Root be $2a$, the Square will be $4a^2$, Cube $8a^3$, *&c.* for the Numbers ſet before the Letters are multiply'd by themſelves, as was ſaid before.

But (2.) if the Root be a form'd Power, then its Index is to be multiply'd by the (Simple) Index of the Power to be conſtituted; that is, to conſtitute the Square it is to be doubled, for the Cube it is to be trebled, *&c.* as if the Root be a^2, its Square will be a^4, Cube a^6, Bi-quadrate (or Square-Square) a^8, *&c.*

Hence we may underſtand, that ſome Indices of Powers are Simple Numbers, as 2, 3, 4, *&c.* others are compounded of the former by Multiplication; ſo the Index of the Square or Second Power is not only 2, but every Number compounded of 2, as 4, 6, 8, *&c.* whence one and the ſame Quantity may be different kinds of Powers; ſo a^6 is not only the ſixth Power, but alſo the third and ſecond, *&c.*

Laſtly, if (3) another Power is to be compounded from two or more Powers ariſing from the ſame Root, it will be done, by adding the Indices of the given Powers, and placing the Sum of them to the Root. *For Example*, Let the Powers a^2 and a^3 be given, the Power compounded of them will be a^5. So of a^3, a^4, a^5, will come a^{12}.

Nor is the Production of Powers from a Compound Root, ſuppoſe a binomial, *&c.* difficult, tho' it require more labour.

For Example.

$$\begin{array}{rll}
\text{Let the Root be} & a+b & \\
 & a+b & \text{multipl.} \\ \cline{2-2}
 & a^2+ab & \\
 & \quad +ab+b^2 & \\ \cline{2-2}
\text{the Square will be} & a^2+2ab+b^2 & \\
 & a+b & \text{multipl.} \\ \cline{2-2}
 & a^3+2a^2b+ab^2 & \\
 & \quad +a^2b+2ab^2+b^3 & \\ \cline{2-2}
 & a^3+3a^2b+3ab^2+b^3 & \text{Cube} \\
 & a+b & \\ \cline{2-2}
 & a^4+3a^3b+3a^2b^2+ab^3 & \\
 & \quad +a^3b+3a^2b^2+3ab^3+b^4 & \\ \cline{2-2}
 & a^4+4a^3b+6a^2b^2+4ab^3+b^4 & \text{Biquadrate.}
\end{array}$$

And after this manner you may frame different Tables of Powers from Compound Roots, as far as you pleaſe.

But the Powers produc'd from any Compound Root, according to their ſeveral Orders and Degrees, have certain Properties, both in reſpect of the Parts themſelves of which they are compounded, and their Generations; as alſo in reſpect of the Signs + and —. Of which, and other things of the ſame kind relating to them, we refer to *Oughtred's Clavis.*

So much for the *Geneſis* of Powers; which, if well underſtood in the Literal Practiſe (uſing the Figures as far as poſſible) will make the way eaſie to the *Analyſis*; which is nothing but the finding out or extracting the Root from any propos'd Quantity: The ſame will likewiſe, in a compendious way, ſerve inſtead of a Rule for the Practiſe of Numbers.

EXTRACTION Of the SQUARE-ROOT.

[*Whoſe Sign is* √*q*, *or* √ *alone.*]

(1.) *From Simple Quantities.*

Which is done by dividing the Index of the Power propos'd by 2, and if the Numerical Square be prefix'd, by extracting the Root ſeparately from that Square.

$$\text{From}\left\{\begin{array}{l}a^2\\ b^2\\ c^2\end{array}\right.\text{the Sq. Root is}\left\{\begin{array}{l}a\\ b\\ c\end{array}\right.\text{from}\left\{\begin{array}{l}4a^2\\ 9b^2\\ 16d^2\end{array}\right.\surd\text{q is}\quad\begin{array}{l}2a\\ 3b\\ 4d\end{array}$$

$$\text{So from}\left\{\begin{array}{l}a^2b^2\\ c^4\\ 25d^6\end{array}\right.\text{the Sq. Root is}\quad\begin{array}{l}ab\\ c^2\\ 5d^3\end{array}$$

(2.) *From Compound Quantities.*

Where the Practiſe is thus:

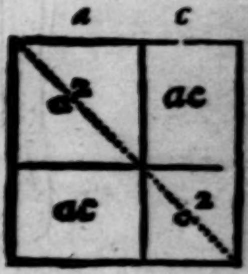
$$\begin{array}{rl|l}\text{Extr. }\surd q.\text{ from} & a^2+2ac+c^2 & a+c\text{ the Root fou.}\\ & a^2+2a\ +c & \\ & \underline{c}\quad\ \underline{\text{mult.}} & \\ & 2ac+c^2\ \text{ſubtr.} & \end{array}$$

1.) Let us ſuppoſe the Root of the Diagonal Square a^2, which is a, and take away the Square itſelf a^2 by ſubtraction.

2.) Double the firſt Number a of the Root found, the double will be $2a$, and write this under $2ac$ (the ſum of the complements) divide [$2ac$ by $2a$] the quotient is c, and will be the other Member, both of the Root and alſo the Diviſor, which was hitherto imperfect, ſo that the whole Diviſor is $2a+c$: Which muſt be multiply'd into its quotient c, and the product ſubtracted from what ſtands over it; (3.) ſince nothing remains at laſt, it is a proof of a perfect Square, and that the Root is truly found.

[*Note.* The doubling of the Root is uſually call'd the formation of the requiſite Diviſor; as in the extraction of the Cube Root the triplication of the Square, *&c.*]

3.) And further, if there be yet more Members of the Root to be found, [as in the laſt Example] all the foregoing Members are to be doubled; and the Diviſion to be perform'd in the ſame manner to the end, till nothing remains.



The reaſon of the operation is eaſily manifeſt from the Geneſis it ſelf, and the annex'd Figure. For the Diagonal Square a^2 gives the firſt member of the Root, which being taken away, the Gnomon remains, conſiſting of the double Complement ac, and the other Diagonal Square c^2. Since therefore the complements are produc'd by the multiplication of the two parts of the Root into one another [that is, of a into c] it follows, that if the Sum of the complements $2ac$ be divided, by the firſt member of the Root doubled as [$2a$] that then the Quotient will be the other member of the Root, *viz.* [c]. Which though it preſently appears in the literal Practiſe from the other Diagonal Square, yet in the Practiſe of Numbers it is to be found out by the means of the ſaid Diviſor [$2a$]; tho' this Diviſor be imperfect till the Quotient it ſelf be join'd to it.

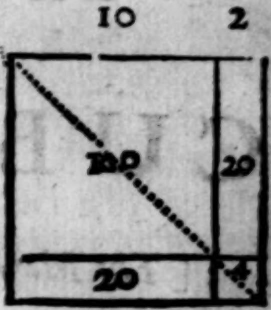
§. From $\cancel{a^2}-2ab+b^2$ | $a-b$ √ ſought.
$\cancel{a^2}+2a\ -b$
$-b$ mult.
$-2ab-b^2$ ſubtr.

§. From $4a^2+8ab+4b^2$ | $2a+b$ √ ſought.
$4c^2-16cd+16d^2$ | $2c-4d$ √ ſought.

§. From $b^2+2bc+c^2+2bd+2cd+d^2$ | $b+c+d$ √ ſought.
$\cancel{b^2}+2b\ +c$ mult.
c
$+2bc+c^2$ ſubtr.
$+2b\ +2c\ +d$ mult.
$+\ d$
$+2bd+2cd+d^2$ ſubtr.

The

The Application of what has been ſaid in Numbers.



Extract √*q* from 144 which is the number 12, let this be conceiv'd to be divided ſo, that the firſt number (1) ſtanding for [10] may be reckon'd for one Member, and the Number [2] for the other, and then proceed as above.

144 | 10+2 that is 12 √ ſought.
100   10
44   2
20   12
2
44

But the ſame may be done (tho' not with equal convenience) by dividing the Root 12 into what parts ſoever, ſuppoſe into 9 and 3, into 8 and 4.

EXTRA-

EXTRACTION
Of the
CUBE-ROOT.

[*The Mark of which is* √ cub. *or* √ *c.*]

(1.) *From ſimple Quantities.*

Which is done by dividing the Index of the Power propos'd by 3, and if a Cube Number be prefix'd, the Cube-Root ought to be extracted by it ſelf.

$$\text{From}\left\{\begin{array}{l} a^3\\ b^3\\ 8c^3\\ 27d^3\end{array}\right.\text{the Cube-Root is}\left\{\begin{array}{l} a\\ b\\ 2c\\ 3d\end{array}\right.$$

$$\text{From}\left\{\begin{array}{l} a^3d^3\\ 125b^3c^3\\ 64b^6\end{array}\right.\text{the Cube-Root is}\left\{\begin{array}{l} ad\\ 5bc\\ 4b^2\end{array}\right.$$

(2.) *From Compound Quantities.*

Where the Practiſe is thus:

From $\sqrt{c}$. from $\cancel{b^3}+3b^2c+3bc^2+c^3 \mid b+c$ $\sqrt{c}$ ſought.

$$\begin{array}{lcc|c} & \cancel{b^3}+3b^2 & +3b & \\ \text{mult.} & (\;+c & +c^2 & +c^3 \\ \hline & +3b^2c & +3bc^2 & +c^3 \end{array}$$

1.) Put down the Cubical-Root of the Diagonal Cube b^3, which is b, and take away the Cube it ſelf b^3 by Subtraction.

2.) Let the firſt member b of the Root which is found, be trebled, and write down the triple $3b$ under $3bc^2$, as the correſpondent term or complement. Then multiply the triple $3b$ by the ſaid member b of the Root, the Product will be $3b2$, which write under the correſpondent complement $3b^2$; ſo you will have part of the Diviſor. Therefore divide $3b^2c$ by $3b^2$, the quotient c will be c, the other member of the Root: which being multiply'd both into the triple of the preceding member b, ſo as to make $3bc$, and alſo into it ſelf ſo as to make the Sqaure $c2$, there will ariſe the whole Diviſor $3b2+3bc+c^2$, which at laſt may be multiply'd into its quotient c, and the product muſt be ſubtracted from the above-written: [Or which will be all one, let the member c of the Root which is found, be multiply'd into $3b2$, and its Square $c2$ into $3b$, to which laſtly joyn its Cube c^3].

3.) If more members of the Root are yet to be found out, let all the found members be trebled, and write that triple down in its proper place, then multiply the ſame treble by all the known Members of the Root, and write the Product under it, according as the Letters themſelves direct; then divide, *&c.* as above *Num.* 2. by repeating the very ſame Operation, as there ſhall be occaſion, to the very end; where, when nothing remains, it is a ſign of a perfect Cube, and that the Root was truly found, the proof of which will be had by multiplying the Root found Cubically.

The reaſon of the Operation is plain from the *Geneſis* and Application of a ſolid Figure. For the Diagonal Cube b^3 gives the firſt Member of the Root, which being taken away the Gnomon remains, conſiſting of three complements b^2c [equal and alike] and likewiſe of three bc^2; and alſo of another Diagonal Cube c^3. Since therefore the firſt order of complements [b^2c] is made by the multiplication of the Square of the firſt member of the Root [b] and under the ſecond member c; it follows that if the Sum of theſe complements [$3b^2c$] be divided by $3b^2$, that is, by the triple of the Root b multiply'd into the Root b, or which is the ſame, by the trebled Square of the Root, that then c the other Member of the Root will be the Quotient.

§ From $a^3-3a^2b+3ab^2-b^3$ | $a-b$ $\sqrt{c}$ ſought.

$$\begin{array}{llll} & a^3+3a^2 & +3a & \\ \text{mult.} & -b & +b^2 & \\ \hline & -3a^2b & +3ab^2 & -b^3 \end{array}$$

§ From $8c^3-12c^2d+6cd^7-d^3$ | $2c-d$ $\sqrt{c}$ ſought.

$$\begin{array}{llll} & 8c^3+12c^2 & +6c & \\ \text{mult.} & (-d & +d^2 & \\ \hline & -12c^2d & +6cd^2 & -d^3 \end{array}$$

The Application in Numbers.

Suppoſe the Number 12167 were given whoſe Cube Root is 23, let this be conceiv'd to be divided that the firſt Member (2) repreſenting tens, may conſtitute one part [20] and the other (3) the remainder; that is, the Root 23 may be divided into 20+3, which being ſuppos'd, proceed as follows.

ſubtr.

$$\begin{array}{r r l}
 & 12167 & \left\{\begin{array}{l} b+c \\ 20+3 \end{array}\right. \\
\text{ſubtr.} & \cancel{8}\cancel{0}\cancel{0}\cancel{0} & \\
\hline
 & 4167 & \text{mult.} \\
 3\cancel{b} & \cancel{6}\cancel{0} & \\
 \text{add.}\left\{\begin{array}{r} 3b^2 \\ 3bc \\ c^2 \end{array}\right. & \begin{array}{r} 1200 \\ 180 \\ 9 \end{array} & \begin{array}{l} 20=b\ 60=3b \quad 3=c \mid 180=3bc \\ 400=b^2\ 1200=3b^2 \mid 9=c^2 \end{array} \\
\hline
 & 1389 & \text{the whole Diviſor.} \\
 & \text{mult. } 3 & \text{Quotient.} \\
\hline
\text{ſubtr.} & 4167 &
\end{array}$$

EX-

EXTRACTION
Of the
Biquadrate ROOT.

[*Whoſe Mark is this*, $\sqrt{qq}$, *or* w]

(1.) *From Simple Quantities.*

Which is done by dividing the Index of the propos'd Power by 4, and if a Bi-quadrate in Numbers be preſix'd, let the Bi-quadratick Root of it be ſeparately extracted.

$$\text{From}\begin{cases} a^4 \\ b^4 \end{cases} \sqrt{qq}\ \text{is} \begin{cases} a \\ b \end{cases} \text{from} \begin{cases} b^4 \\ d^4 \end{cases} w\ \text{is}$$

(2.) *From Compound Quantities.*

Where the Practiſe is thus:

$$\begin{array}{lllll}
\text{1.) From } \cancel{b^4} + 4b^3c + 6b^2c^2 + 4c^3b + c^4 \,|\, b + c\ w \text{ ſought:} \\
\cancel{b^4} + \ b^3 \ + \ b^2 \ + \quad b \quad \text{mult.} \\
\qquad\ \ 4 \qquad 6 \qquad\quad 4 \\
\hline
\text{mult.} \quad + 4b^3 + 6b^2 + 4b \\
\qquad\quad\ \ + \ c \ \ + \ c^2 + \ c^3 \\
\hline
\qquad\quad\ \ + 4b^3c + 6b^2c^2 + 4c^3b \ \,|\, \ + c^4
\end{array}$$

1.) Let us ſuppoſe w of the Bi-quadrate of the Diagonal Biquadrate b^4, which is b; and take away that b^4 by Subtraction.

2.)

2.) The Member of the Root b being found, let its Square b^2, and alſo its Cube b^3, be put down in their proper places from the Right Hand to the Left, and let b be always multiply'd by 4, b^2 by 6, and b^3 by 4 again, the Product will be $4b^3+6b^2+4b$, which in part compoſes the Diviſor for the following Member of the Root. Let therefore $4b^3c$ be divided by $4b^3$, the Quotient c will be the other Member of the Root; which being multiply'd into $6b^2$, ſo as to make $6b^2c$, and its Square c^2 into $4b$; ſo as to make $4bc^2$; and alſo into it ſelf cubically, ſo as to make c^3 there will ariſe the whole Diviſor $4b^3+6b^2c+4bc^2$ the whole of which is at laſt to be multiply'd into its Quotient c, and the Product ſubſtracted from that which is written above. Or (which is the ſame thing) having gotten the Terms $4b^3+6b^2+4b$, let $4b^3$ be multiply'd by c, $6b^2$ by c^2, and $4b$ by c^3, that is, the Cube of c by c it ſelf; the Quadratick Term by c^2 the Square of the Root; and the Radical Term by c^3 the Cube of the Root; to which, laſtly, joyn its Bi-quadrate, and the whole Sum will be = to that above.

3.) If more Members of the Root are to be found out, the ſame Operation is to be repeated as in *Num.* 2. by taking all the Members of the Root before found, *&c.* And ſince nothing remains at laſt, it is a Proof of a perfect Bi-quadrate, and that the Root is truly found.

The Reaſon of the Operation will be plain, as in the foregoing; excepting that no Geometrical Figures go beyond the Cubical Power.

$$\text{2.) From}\quad \begin{array}{lllll|l} a^4 & -4a^3b & +6a^2b^2 & -4b^3a & +b^4 & a-b\ \sqrt{qq}. \\ a^4 & +4a^3 & +6a^2 & +4a & & \\ & -b & +b^2 & -b^3\ \text{mult.} & & \\ \hline & -4a^3b & +6a^2b^2 & -4b^3a & & +b^4 \end{array}$$

The Practice in Numbers.

In a like manner alſo will the Practiſe in Numbers proceed, but it will be more eaſily done by a double Extraction of the Square-Root, that is, if the Square-Root be firſt extracted out of the Number propoſ'd, and then the Square-Root of this Square-Root; for this Second Root will be the Bi-quadratic Root of the Number propoſ'd.

For Example, Suppoſe the Number 38416, the Square-Root of that will be 196; and the Square Root of this 14. Therefore 14 will be the Bi-quadratic Root of the Number propoſ'd, 38416.

So much concerning the extraction of Roots from rational Powers, that is, where the above mention'd Rules take place, and leave nothing remaining.

But when the Extraction cannot proceed after the manner aforeſaid, it is a Proof that the Quantity or Power propoſ'd is ſo far irrational, to which therefore we need only prefix the Sign of the Root ſought; and when it is a Compound Quantity, that Sign affects all the Terms, and it is commonly call'd *An Univerſal Root*. And this is the Origin of *Irrational* or *Surd* Quantities, of which in their place. Therefore, *for Example*,

$$\text{The Square-Root}\left\{\begin{array}{c} a^3\\ ab\\ 5b^2c\\ 3c^4 \end{array}\right\}\begin{array}{c}\text{will}\\ \text{be}\end{array}\begin{array}{l} \sqrt{a^3}\\ \sqrt{ab}\\ \sqrt{5b^2c}\\ \sqrt{3c^4} \end{array}$$

[Because the Index [3] cannot be divided by 2.

[In this place indeed the Index [4] may be divided by 2, but the prefix'd Number is not a Square Number.

$$\text{The Cubic Root}\left\{\begin{array}{c} a^4\\ 8a^2b\\ 6c3\\ 5d^2 \end{array}\right\}\begin{array}{c}\text{will}\\ \text{be}\end{array}\begin{array}{l} \sqrt{c.\ a^4}\\ \sqrt{c.\ 8a^2b}\\ \sqrt{c.\ 6c3}\\ \sqrt{c.\ 5d^2} \end{array}$$

$$\begin{array}{c}\text{Likewise the}\\ \text{Universal}\\ \text{Square-Root}\end{array}\left\{\begin{array}{c} a^2+b^2\\ a^2+c^2-b^2\\ 4b^3+bc^2 \end{array}\right\}\begin{array}{c}\text{will}\\ \text{be}\end{array}\begin{array}{l} \sqrt{\overline{a^2+b_2}}\,^{*}\\ \sqrt{\overline{a^2+c^2-b^2}}\\ \sqrt{\overline{4b^3+3bc^2}} \end{array}$$

* That is, add the two Squares a^2 and b^2, and extract the Square Root from the Sum.

$$\begin{array}{c}\text{The}\\ \text{Universal}\\ \text{Cubic-Root}\end{array}\left\{\begin{array}{c} a^3+8b^3\\ 2a^3-b^3\\ 3b^2c+2bc^2-c^3 \end{array}\right\}\begin{array}{c}\text{will}\\ \text{be}\end{array}\begin{array}{l} \sqrt{c.\ \overline{a^3+8b_3}}\\ \sqrt{c.\ \overline{2a3-b3}}\\ \sqrt{c.\ \overline{b^2c+2bc^2-c^3}} \end{array}$$

II. THE OPERATIONS OF FRACTIONS

Where, for Exerciſe ſake, the Practiſe of Numbers and Letters are both deliver'd.

CHAP. I.

The Notation of Fractions.

§. *Proper Fractions.*

The Numerator,	1	2	3	a	$2a$	a^2	Suppoſing b
The Denominator,	2	3	5	b	$3b$	b^2	greater than a

§. *Improper Fractions.*

$$\frac{2}{2} \,\Big|\, \frac{3}{2} \,\Big|\, \frac{9}{4} \,\Big|\, \frac{12}{5} \,\Big|\Big|\, \frac{a}{a} \,\Big|\, \frac{b}{a} \,\Big|\, \frac{3b}{2a} \,\Big|\, \frac{2b^2}{a^2}$$

See alſo *Chap.* II. *Memb.* 1.

Where Note, When the Denominator and Numerator are equal, the Fraction is always equal to 1, *&c.*

§ *Fractions*

§. *Mixt Fractions.*

$3\frac{1}{2} \mid 5\frac{3}{4} \mid 8\frac{5}{8} \parallel a+\frac{a}{b} \mid 2b-\frac{c^2}{3a}$

§. *Fractions of Fractions, or ſecond Fractions.*

$\frac{1}{2}$ of $\frac{3}{4}$ that is, the half of three Fourths, which is uſually written thus, $\frac{1}{2}$, $\frac{3}{4}$, a comma being put between; likewiſe $\frac{2}{3}$ of $\frac{4}{5}$, two thirds of four fifths.

In theſe, the latter Fraction or Term, is the principal and greater, which bears the relation of the whole: But the former Term, which is the leſſer, is referr'd thereto as a part.

So in Letters: $\frac{a}{b}$ of $\frac{2a}{3b}$ and let $a=3$. $b=4$.

CHAP. II.

The Reduction or Tranſmutation of Fractions.

§. Firſt MEMBER.

To reduce improper Fractions to Integers.

DIvide the Numerator by the Denominator, the Quotient will give the Integers, *&c.*

For Example, reduce 1.) $\frac{12}{4}$ | 3 the quotient.

2.) $\frac{8}{3}$ | $2\frac{2}{3}$ the quotient.

3.) $\frac{56}{7}$ | 8 quotient.

4.) $\frac{65}{12}$ | $5\frac{5}{12}$ quotient.

So in Letters: $\frac{2aa}{a}$ | $2a$ quotient | $\frac{6bc}{2c}$ | $3b$ quot. | $\frac{a}{a}$ | 1

§. Second MEMBER.

To reduce Integers into the form of a Fraction.

1.) If there are only Integers given, and no certain Denominator be given, let a Unit be written under the propos'd Number or Quantity, and it will be done.

Suppoſe theſe Integers given,	*The Fractions will be*
6	$\frac{6}{1}$
15	$\frac{15}{1}$
1	$\frac{1}{1}$
a	$\frac{a}{1}$
$2b$	$\frac{2b}{1}$
$3ab$	$\frac{3ab}{1}$

2.) If there are only Integers given, and there be a certain Denominator aſſign'd for the Fraction,

Let the propos'd Quantity be multiply'd by the Denominator given, and let the Denominator be written under the Product.

If the Number 7 be to be reduc'd into a Fraction, whoſe Denominator is 4, it will make $\frac{28}{4}$.

[Which is no more than if I ſhould ſay that ſeven whole Pounds were to be reſolv'd into Farthings.]

Let the quantity b be given, and the Denominator of the Fraction $a - b$, there will ariſe this improper Fraction $\frac{ab - bb}{a - b}$

Let the Quantity $2a + c$ be given, and the Denominator f, there will ariſe this Fraction $\frac{2af + cf}{f}$

3.) If the Integers have a Fraction annex'd to them, let the Integer Quantity be multiply'd by the Denominator of the adherent Fraction; let the Numerator be added to the Product, and the Denominator written under the Sum.

Suppoſe,	$2\frac{3}{4}$	$3\frac{2}{5}$	$9\frac{4}{10}$	$12\frac{6}{25}$
	$\frac{8}{3}$	$\frac{15}{2}$	$\frac{90}{4}$	$\begin{matrix}42\\50\\25\end{matrix}$
They make	$\frac{11}{4}$	$\frac{17}{5}$	$\frac{94}{10}$	$\frac{106}{25}$

Suppoſe $c + \frac{de}{a}$ there will come out $\frac{ac + de}{a}$.

Suppoſe $b + c + \frac{b^2 + c^2}{b - c}$ there will come out $\frac{2b^2}{b - c}$

Suppoſe $2ab - \frac{a^2c}{b}$ there will come out $\frac{2abb - a^2c}{b}$.

§. Third MEMBER.

To reduce Fractions propos'd in greater Terms to leſſer Terms of equal value (when it is poſſible.)

For any one and the ſame Fraction, may be expreſs'd innumerable ways, of which the principal is that which gives it in the leaſt Terms.

For every Fraction is not reducible to leſs Terms, but thoſe only are ſo, in which the Numerator and Denominator have one or more common Diviſor.

If therefore the greateſt common meaſure of both Terms (that is, of the Numerator and Denominator) be inquir'd (as ſhall be ſhewn afterwards) and each be divided by that common meaſure, the Quotients will give the leaſt Terms.

Or which will be all one, divide them by any the ſame Diviſors, which offer themſelves, and leave nothing remaining, till no common Diviſor occur but Unity.

Wherefore in this place (eſpecially as to the practiſe of Numbers) obſerve the following Caſes.

1.) When the Denominator may be exactly divided by the Numerator, then the Numerator it ſelf is the greateſt common Diviſor. And in this Caſe, the Numerator of the reduc'd Fraction is always Unity.

Suppoſe theſe Fractions.

$$\frac{6}{24} \quad \frac{5}{15} \quad \frac{8}{40} \quad \frac{4}{32} \quad \frac{7}{63}$$

Equivalent to theſe in the leaſt Terms, there will be

$$\frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{5} \quad \frac{1}{8} \quad \frac{1}{9}$$

But if it does not immediately appear, as when the Numbers are greater, then we muſt proceed according to the Rules which ſhall be given after.

2.) If both Terms are *even* Numbers, you muſt halve them as far as you can.

Let the Fraction $\frac{32}{64}$ be given, by continually halving will ariſe theſe $\frac{16}{32}$ | $\frac{8}{16}$ | $\frac{4}{8}$ | $\frac{2}{4}$ | $\frac{1}{2}$ every one of which are equivalent to the Fraction given; but of theſe $\frac{1}{2}$, as conſiſting of the leaſt Terms, is the moſt known.

Likewiſe let the Fraction $\frac{16}{24}$ be given, if we halve them, theſe will be equivalent to them, $\frac{8}{12}$ $\frac{4}{6}$ $\frac{2}{3}$.

3.) If both the Terms have on the right Hand one or more Cyphers, let the ſame Number of Cyphers be ſtruck off from both [which is no more than to divide them both by 10 or 100] *&c.*

Let $\frac{30}{40}$ or $\frac{300}{400}$ or $\frac{3000}{4000}$ be given, caſt away as many Cyphers from one ſide as the other, and then they become $\frac{3}{4}$.

Let ,$\frac{400}{7000}$ be given, and there comes out firſt $\frac{4}{70}$, and then taking the halves, it is $\frac{2}{35}$.

4.) If the Terms are otherwiſe, and the Numbers not very great, it will eaſily appear [from the *Pythagoric Table*] whether they have one or more common Diviſors, with which work as before.

Let the Fraction $\frac{42}{63}$ be given; here firſt both Terms may be divided by 7, from whence ariſe $\frac{6}{9}$, and again both may be divided by 3, ſo that it makes $\frac{2}{3}$.

Likewiſe let the Fraction $\frac{15}{35}$ be given, then dividing both ſides by 5, $\frac{3}{7}$ will ariſe.

So if the Fraction $\frac{48}{54}$ be given; dividing by 6, the equivalent will be $\frac{8}{9}$.

5.) If it cannot be done by the foreſaid means, as when the Numbers are very large, then let the greateſt common Meaſure be inquir'd after this manner.

Let the greater Term always be divided by the leſs, and let it be ſo in all the Remainders, as well as in the Numbers firſt given; that is, let every foregoing Diviſor be divided by the correſpondent Remainder, taking no notice of the Quotients: The laſt Diviſor, which leaves nothing remaining, will be the greateſt common Meaſure of both. [*See the Example* A.]

But if at laſt Unity remains, it is a Proof that the Terms of the Fraction propos'd have no common Diviſor but Unity, and therefore that Fraction cannot be reduc'd to leſſer Terms. [*See Example* B.]

(12 rem.
A] Let the Fraction be given $\frac{105}{432}$ | Div. ~~432~~
1) ~~105~~ | 4

(9 | (3 | nothing remains.
2) ~~105~~ | 8 | 3) ~~12~~ | 1 | 4) 9 | 3
~~12~~ | ~~9~~ | 3

and the laſt Diviſor was 3; this therefore is the greateſt common Diviſor for reducing the Fraction given, as follows.

The

The Numerator.		*The Denominator.*		*The Fraction reduc'd.*
Div. $\frac{\cancel{105}}{\cancel{3}}$	35	$\frac{\cancel{432}}{\cancel{3}}$	144	$\frac{35}{144}$

B] Suppoſe moreover the Fraction $\frac{35}{144}$ be given, to be reduc'd to its leaſt Terms,

Div. 1.) $\frac{\cancel{144}}{\cancel{35}}$ (4	4	2.) $\frac{\cancel{35}}{\cancel{4}}$ (3	8	3.) $\frac{\cancel{4}}{\cancel{3}}$ (1	1

whence it is plain, that this Fraction $\frac{35}{144}$ is already in its leaſt Terms, and admits of no further Reduction.

§. In the Literal Practiſe, if the Quantities are not too much compounded, the common Diviſor (if it has any) is eaſily known (eſpecially to one well vers'd in the Practiſe of Multiplication and Diviſion) which being had, proceed the ſame way as above in Numbers; that is, dividing both Terms by the greateſt common Diviſor found, and taking the Quotients.

But when the Quantities are very large, and much compounded, the diſcovery of a common Diviſor is a more laborious and tedious Operation, than is proper to be deliver'd in this place.

Wherefore the Examples following will be ſufficient.

Propos'd Fractions.	Common Diviſors.	Reduc'd Fractions.
$\frac{abc}{bd}$	b	$\frac{ac}{d}$
$\frac{4a^2b^2}{2abc}$	$2ab$	$\frac{2ab}{c}$
$\frac{abd^2}{b^2d}$	bd	$\frac{ad}{b}$
$\frac{2a^3b^4c}{6ab^2c}$	$2ab^2c$	$\frac{a^2b^2}{3}$
$\frac{b^2c+b^2d}{cd+d^2}$	$c+d$	$\frac{b^2}{d}$
$\frac{a^2+2ab+b^2}{a^2-b^2}$	$a+b$	$\frac{a+b}{a-b}$
$\frac{f^2c-f^2d-c^3+c^2d}{dc-d^2}$	$c-d$	$\frac{f^2-c^2}{d}$
$\frac{c^2-d^2}{bc+bd+c^2+cd}$	$c+d$	$\frac{c-d}{b+c}$
$\frac{a^3+b^3}{a^2-b^2}$	$a+b$	$\frac{a^2+b^2-ab}{a-b}$

OF

OF THE Invention, Diſtinction, Reduction and Reſolution of *Algebraic Æquations.*

WHen any Problem or Queſtion is propos'd, imagine the thing requir'd to be done, and inſtead of the *Sought* Quantity (whether it be Number or Magnitude) put x, y, or z, that is, one of the laſt Letters of the Alphabet; but for the *Given* Quantities the firſt Letters, as a, b, c, *&c.* by which means the *Given* Quantities will be more eaſily diſtinguiſh'd from thoſe that are *Sought*.

Then let the Quantities as well *Given* as *Sought* be compar'd, according to the conditions of the Queſtion, by Addition, Subſtraction, Multiplication and Diviſion, till at laſt ſomething be found equal to the Quantity *Sought*, or to ſome Power of the ſame.

But becauſe in almoſt all Equations,as they firſt ariſe from the conditions of the Queſtion known and unknown, Quantities are variouſly mingled and confounded together: Therefore the Terms of every Equation are to be ſo order'd and reduc'd, that the *Given* may poſſeſs one part, and the *Sought* another. But then the Quantities on both ſides are to be reduc'd into the moſt ſimple Form and Expreſſion of a Root; which we call reduction of Equations to the moſt ſimple Form: And the principal Rules of this are as follows.

If Equal things are added to or ſubtracted from Equal things, the Sums and Remainders will be Equal.

If Equal things are multiply'd or divided by Equal things, what ariſes will be Equal.

The Homogeneal Roots of Equal things will be Equal.

Further, Equations are of ſeveral kinds, or orders, *viz.* as many as there are degrees of Powers or dimenſions in the progreſſional Scale of Quantities. Whence we have Equations,

1. *Simple,* } { 3. *Cubical,*
2. *Quadratic,* } { 4. *Bi-quadratic,* &c.

The handling of each of which are accordingly conſider'd in the ſeveral diſtinct parts of Algebra.

The reduction of Equations (eſpecially ſimple Ones) is perform'd by the following Operations,

1. *By Addition.*

Let $x - a = b + c$ $\qquad c - x = 0$
add. $+a \qquad +a$ $\qquad +x \qquad +x$
it will be $x = b + c + a$ $\qquad c = x$
Let $Q - b = 0$
$+b \qquad +b$
$Q = b$

2. *By*

2. *By Subſtraction.*

$$\begin{array}{lcl} \text{Let } Q. + \cancel{a} & = & b + c \\ \text{ſubſt. } \quad + \cancel{a} & & + a \\ \hline \text{it will be } Q. & = & b + c - a \end{array} \qquad \begin{array}{rcl} x + a - b & = & 2x + d \\ x \text{ ſubſt.} & & \\ \hline a - b & = & x + d \\ \text{ſubſt. } + d & & + d \\ \hline a - b - d & = & x \end{array}$$

Note. Inſtead of Addition and Subſtraction we may more compendiouſly uſe the tranſpoſition of Quantities from one ſide of the Equation to another, always giving them the contrary Signs.

$$\begin{array}{lcl} \text{Let } a - b - x & = & b + x \\ \text{it will be } a - 2b & = & 2x \end{array}$$

3. *By Multiplication.*

$$\begin{array}{ll} \text{Let} & \left(\begin{array}{c} \frac{x}{a} = b \\ a \text{ mult. } a \end{array} \right) \\ \hline \text{it will be} & \frac{\cancel{a}x}{\cancel{a}} = ab \\ \text{that is} & x = ab \end{array} \qquad \begin{array}{l} \frac{Q.}{3b} = a - c \\ \\ \text{mult. by } 3b \\ Q. \text{ will be} = \text{to } 3ab - 3bc \end{array}$$

4. *By Diviſion.*

$$\begin{array}{ll} \text{Let} & x^2 = 6x \\ & x \text{ div.} \\ \hline \text{it will be} & x = 6 \end{array} \qquad \begin{array}{c} 3x^2 = 9x \\ 3x \text{ div.} \\ \hline x = 3 \end{array}$$

Let $x^2 + ax = bx + cx$
x div.
it will be $x + a = b + c$
and $x = b + c - a$

5. *By Extraction of Roots.*

Let $x^2 = a^2$
extr. $\surd$
$x = a$

Let $y^2 = ab$
extr. $\surd$
$y = \sqrt{ab}$

Let $x^2 + ax + \frac{1}{4}a^2 = b^2$
extr. $\surd$
$x + \frac{1}{2}a = b$
and $x = -\frac{1}{2}a + b.$

Let $x^3 - 3bx^2 + 3b^2x - b3 = c3$
extr. $\surd c.$
$x - b = c$
and $x = b + c.$

Problems or *Queſtions,*

Which produce Simple Equations.

1. TO find a Number, which being multiply'd by 3, ſubſtracting 5, from the Product, and the remainder divided by 2, if the Number ſought be added to the Quotient, that the Sum may be 40.

Suppoſe the Number ſought $=x$;

1. *The Practiſe in Numbers will be thus:*

The Number ſought $=x$
multiply'd by 3
the Product will be $3x$
from which ſubſtract 5
the rem. will be $3x-5$
which divide by 2
the Quotient will be $\frac{3x-5}{2}$
add to this the Numb. ſought x
the Sum will be $\frac{3x-5}{2}+x=40$
which by mult. will be $3x-5+2x=80$
that is $5x-5=80$
and by tranſpoſing $5x=80+5$
that is $5x=85$
and by dividing by 5
$x=17=$ the Number ſought.

§ *Proof.*

The Number found 17
multipl. by 3
51
ſubtr. 5
46
Divide by 2
23
add 17
the Sum is 40 which was the Number propos'd.

2. *The Practiſe in Letters.*

Let 3 or the multiplying Number $=a$
5 to be ſubſtracted $=b$
2 the Diviſor $=c$
40 the Sum $=d$

And all theſe will be given or may be aſſumed at pleaſure.

Now then, the Number ſought $=x$
multiply by a
the product ax
ſubtr. b
rem. $ax-b$
divide by c
quotient $\frac{ax-b}{c}$
add x
the Sum $\frac{ax-b}{c}+x=d$
therefore by multiplying $ax-b+cx=cd$
and by tranſpoſing $ax+cx=cd+b$
and dividing by that $a+c$

$$x=\frac{cd+b}{a+c}$$

From hence comes the general Rule for ſuch ſort of Queſtions: Let the given Number of the Equation be multiply'd by the given Diviſor: Let the Number to be ſubſtracted be added to the Product: Divide the total by the Multiplier, augmented by the Diviſor; the Quotient will give the Number ſought.

§. Where we muſt obſerve, that the Invention of the thing ſought proceeds by an Operation or Method contrary to that of the Queſtion. And ſo in other Caſes.

2. To find a Number, which being multiplied by 12, and 48 added to the Product, as much may be produc'd, as if the ſame Number ſought were multiply'd by 18. Nº = 8

Or generally thus,

To find a Number, which being multiply'd by *a*, and *b* added to the Product, as much may be produc'd, as if the ſame Number were multiply'd by *c*.

3. To find a Number, to which if 11 be added and 7 ſubſtracted from the ſame Number [*viz.* the firſt] the Sum of the Addition may be double the remainder. Nº = 25

Or in general.

To find a Number, to which if *b* be added, and *c* ſubſtracted from the ſame Number [*viz.* the firſt] the Sum of the Addition may be to the remainder in a given *ratio*, *viz.* double, treble, *&c.*

4. To find a Number, to which if its double [treble, quadruple, *&c.*] be added, the Square of the ſame Number may be produc'd.

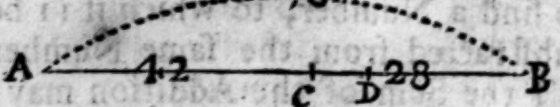
Nº = 10

5. To find a Number, which if added to it ſelf, and the Sum multiply'd by the ſame; and the ſame Number ſtill ſubſtracted from the Product: And laſtly, the remainder divided by the ſame, that it may produce 13. N° = 7

6. To divide the Number 16 into 2 parts, ſo that the Square of the greater part may exceed the Square of the leſs by 32. N° = 9+7=16

7. To divide the Number 36 into two parts, ſo that if 12 be added to the firſt, and 6 to the ſecond, the former may be the double of the latter. N° = 12, 24

8. Let the Line AB (of 70 parts) be divided any how in C [ſo that AC may be 42, BC 48] it is required to divide the ſame Line again in another Point: *For Example*, in D, ſo that the Rectangle ADC may be equal to the Square DB. Let the Segment CD be inquir'd [which being obtain'd, AD, DB will be known].



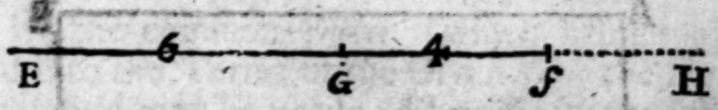
c.D = 8

§. *Or thus in Numbers.*

Suppoſe two Numbers 42 and 28; a third is to be found, which being added to 42, if the whole be multiply'd by the Number ſought, as much may be produc'd, as if the ſame Number ſought, being ſubſtracted from 28, the Remainder was ſquar'd.

AD = 50, CD = 8, DB = 20
8 20
400 400

9. Let the Line EF be divided any how in G, [ſo that EG may be 6, GF 4] it is requir'd to produce this right line EF [for Example unto H] ſo that the Rectangle EHF may be equal to the Square GH; the length of FH is requir'd.



N° 8

§. *Or thus in Numbers.*

Let there be two Numbers 6 and 4; we are to find a third, which being added to the Sum of two given Numbers, if the whole be Multiply'd by the Number Sought, as much may be produc'd, as if the Sum of the Number Sought, and the leaſt of the Given Numbers were ſquar'd. N° = 8

10. A General diſpoſing his Army into a Square Battle, finds he has 284 Souldiers over and above; but increaſing each ſide with one Souldier, he wants 25 Souldiers to fill up the Square: How many Souldiers had he?

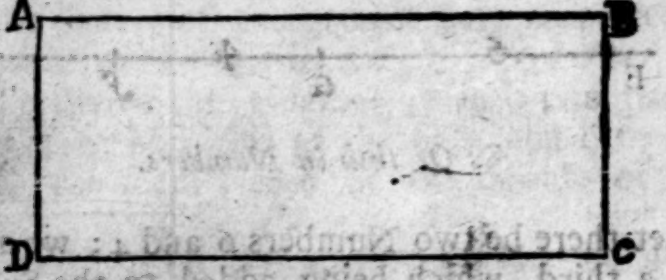
649 — 29 = 24620 Solids

11. A certain Captain ſends out $\frac{1}{3}$ of his Souldiers + 10, there remain $\frac{1}{2}$ + 15. How many Souldiers had he? N° = 140

12. There is an Army to which if you add $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{3}{4}$ of it ſelf, and take away 5000, the Sum Total will be 100000. What was the Number of the Army?

N° = 40645 $\frac{5}{31}$

13. In the Rectangle ABCD, the difference of the greater ſide AB, and of the leſſer ſide BC, is 12, but the difference of the Squares of the ſides 1680. What are the ſides of the Rectangle ABCD?

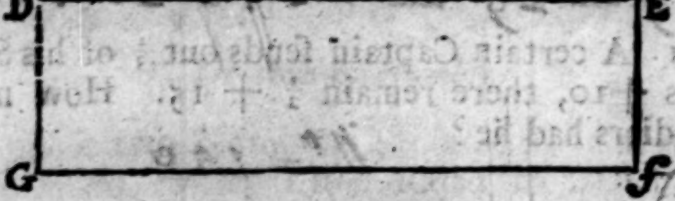


That is, propoſing it generally.

The difference of the Sides, and the difference of the Square of the Sides being Given in a Rectangle, to find the Sides of the Rectangle.

AB = 76, BC = 64

14. The length DE of the Rectangle DEFG, is twice the breadth EF; and the Sum of the Squares of the length and breadth is ten times the Sum of the two Sides DE, EF. What are the Sides of the Rectangle DEFG?



Or generally.

In a Rectangle the *Ratio* of the Sides, and the *Ratio* of the Sum of the Squares of the Sides to the Sum of the Sides themſelves being Given, to find the Sides of the Rectangle.

15. To

DE = 12 EF = 6

15. To find two Numbers in the proportion of 2 to 3, whoſe Product, if they be multiply'd by one another, ſhall be 54.

Let the leſs Number = X

$\because 2 \div 3 : X \mid \frac{3x}{2} =$ the greater Number, or, let the common *Factor* [which being multiply'd into the Terms of the Given *Ratio* the Numbers deſir'd may be produc'd] = X : ∵ the leſs of the Numbers Sought = 2 X
and the greater = 3 X. R° = 6×9 = 54

16. To find two Numbers whoſe *Ratio* is to one another as 4 to 5 : and the Sum of the Squares of both, is 2624. R° = 32, 40

17. To find the ſide of a Square, whoſe *Area* is to the Sum of the ſides in a Given *Ratio*, as 45 to 12 ∴ 225 : 60

18. To find the ſide of a Cube, whoſe *Superficies* is to the Solidity in a Given *Ratio*, as 6 to 11 ∴ $3\frac{25}{216}$: $6\frac{95}{216}$

19. A certain Man hires a Labourer, on this
condition, that for every Day he work'd he
ſhould receive 12 Stivers, but for every 3*s.*
Day he was idle, he ſhould be mulct'd 8 2*s.*
Stivers. When 390 Days were paſt, neither of them were indebted to one another. How many Days did he work ; and how many was he idle ? Plaid 234, work 156

20. A certain Gentleman hires a Servant, and
promiſes him 24 Florins yearly Wages, 24*l.*
together with a Cloak. At 8 Months end
the Servant obtains leave to go away, and inſtead

13*l.* of his Wages receives a Cloak + 13 Florins. How much did the Cloak coſt?

Cloak = 9*l.*

21. A Perſon being ask'd how Old he was, anſwer'd, If I quadruple $\frac{2}{3}$ of my Years, and add $\frac{1}{2}$ of them + 50 to the Product; the Sum will be ſo much above 100, as the number of my Years is now below 100. = 2.36 Years, ~~[illegible]~~

22. One being ask'd what hour of the Day it was? anſwer'd, The Day at this time is 16 hours long; if now $\frac{1}{2}$ of the hours paſt be added to $\frac{2}{3}$ of the remainder, you will have the hour deſir'd, reckoning from Sun-riſing. 9 $\frac{1}{7}$ past 6 $\frac{6}{7}$ to Come

23. From *Norimberg* to *Rome* are 140 Miles: A Traveller ſets out at the ſame time from each of the two Cities, one goes 8 Miles a Day, the other 6. In how many Days from their firſt ſetting out will they meet one another, and how many Miles did each of them go?

meet in 10 Days, first 80, 2d 60 = 140

24. A certain Meſſenger goes 6 Miles every Day: 8 Days after, another follows him, and he goes 10 Miles a Day. In what number of Days will he come up to the firſt? in 12 ~~4~~ Days

25. A certain Meſſenger goes 6 Miles a Day: And after he has gone 56 Miles, another follows him who goes 8 Miles a Day. In how many Days will he come up to him? in 28 Days

26. One bought 3 Books, whoſe Prices were in proportion, as 12, 5, 1: If the Price of the firſt be doubled, of the ſecond trebled, of the third quadrupled; the Sum of theſe Products will as much exceed 10 Imperials, as the Sum of the Prices of the greateſt and middle is below 5. How much did the ſaid Books coſt?

4.14 : 3 : 1.15 : 12 : [illegible] | 14 [illegible] 10 [illegible]

[illegible]

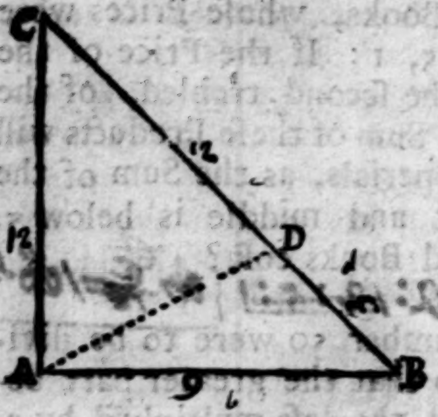
27. Suppoſe the Number 50 were to be divided into two parts, ſo that the greater part being divided by 7, and the leſs multiply'd by 3, the Sum of this Product and the former Quotient may make the ſame Number propos'd, which was 50. n° 35 ÷ 7 = 5, 15 × 3 = 45

28. Let the Number 20 be divided into two parts, ſo that the Square of the leſs part being taken out of the Square of the greater, may leave the very Number propos'd, which was 20 [or may leave the double, treble, *&c.* of the Number propos'd]. n° = 9.5 & 10.5

29. If a Man gains 30 Imperials a Week; how much muſt he ſpend a Week to have 500 Imperials, together with the expence of 4 Weeks remaining at the Years end?

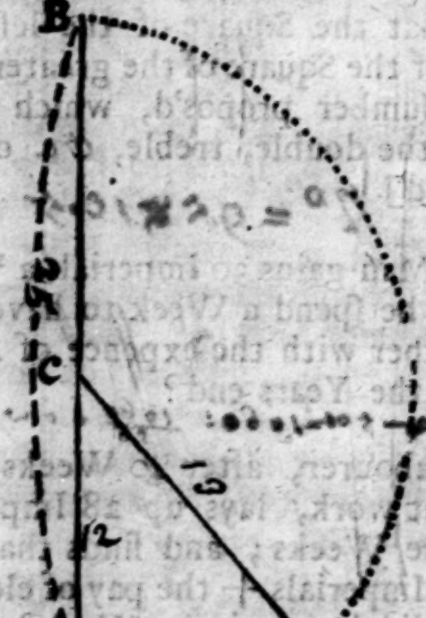
x 52 = 1560 − 500 = 1060. $\frac{1060}{56}$ = one weeks expence = 19 for

30. A Labourer, after 40 Weeks in which he had been at work, lays up 28 Imperials — the pay of three Weeks; and finds that he had expended 36 Imperials + the pay of eleven Weeks. What Pay did he receive a Week? = 2 Imperialls



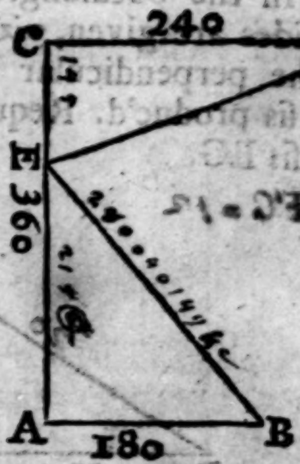
31: In the Rectangle Triangle ABC, is given the baſis AB=9 and the difference of the other Sides, that is, the Segment BD,=3. Requir'd, the Sides AC, BC.

32. In the Rectangle Triangle ABC, is given the Baſis AB=5, and the Sum of the other Sides AC+BC=25. Required, the Sides AC, BC ſeverally.



33. Let

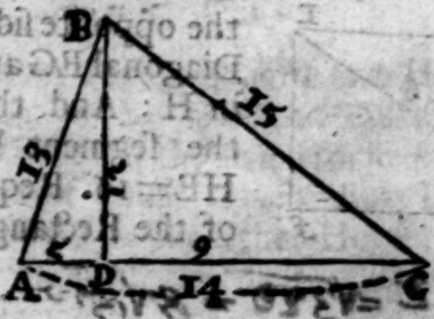
33. Suppoſe two Towers, AB 180 feet high, and CD 240, at the diſtance AC 360 feet. A Ladder is to be ſet upon the line AC, at ſome point, ſuppoſe in E, of ſuch a length, as from thence it may reach the top of both the Towers. We require the point E in the line of diſtance, as alſo the length of the Ladder EB, ED.



§. *That is, in general Terms.*

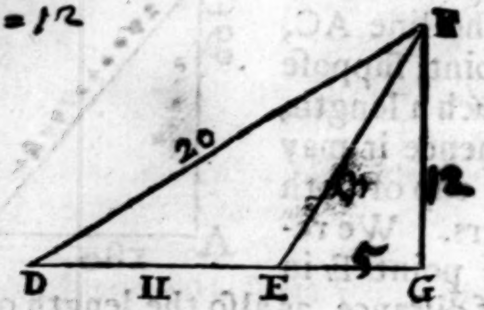
Of two right angled Triangles, which have equal Hypothenuſes, the ſum of the Baſes, and the height of each is given. Requir'd the ſeveral Baſes with their common Hypothenuſe.

34. In the Triangle ABC, the ſeveral ſides AB=13, AC=14, BC=15 are given; and the perpendicular BD being drawn. Requir'd the ſegments of the Baſis AD, DC.

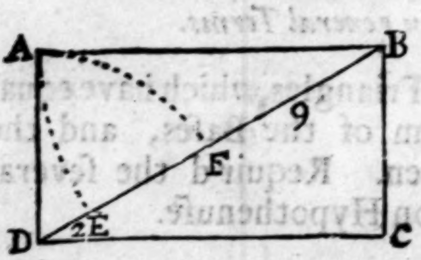


35. In the obtuſangle Triangle DEF the ſeveral ſides are given, *viz.* DE 11, EF 13, DF 20; and the perpendicular FG, being let fall upon the Baſis produc'd. Requir'd the prolongation of the Baſis EG.

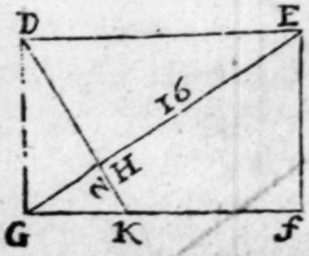
[illegible] = 5, F'G = 12



36. In the Rectangle ABCD, is given the difference between the Length AB and the Diagonal BD, that is, DE=2; and likewiſe the difference between the Breadth AD and the Diagonal BD, that is, FB=9. Requir'd the ſides of the Rectangle AB, AD. = 8 . AB = 15

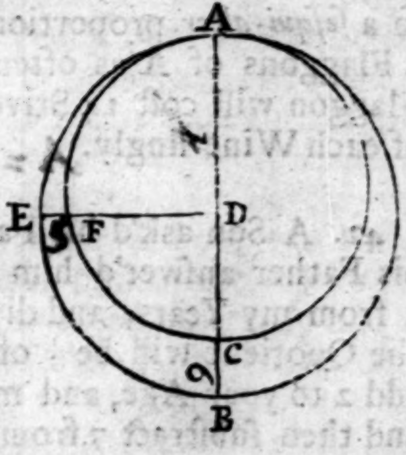


37. In a Rectangle DEFG, the right line DK is drawn from the Angle D to the oppoſite ſide, cutting the Diagonal EG at right Angles in H: And there is given the ſegment HK=2, and HE=16. Requir'd the ſides of the Rectangle.



DG = $\sqrt{80}$, DE = $\sqrt{320} = 8\sqrt{5}$

38. Let there be a Circle, whoſe Diameter is AB, which another leſs Circle whoſe is AC, Diameter touches within in A: and from the Center of the greater Circle D, draw the radius DE at right Angles to AB, cutting the Periphery of the leſſer Circle in F. Now there is given BC [the difference of the Diameters] =9, with the Segment EF=5. Requir'd the Diameters AB, AC of the ſaid Circles. AB = 50, AC = 41



39. Two Companions have got a parcel of Guinea's; ſays A to B, if you will give me one of your Guinea's, I ſhall have as many as you will have left. Nay, replies B, if you will give me one of your Guinea's, I ſhall have twice as many as you will have left. How many Guinea's had each of them? A = 5 B = 7

40. A certain Perſon bought two Horſes, with the Trappings, which coſt 100 Florins; which Trappings if laid on the firſt Horſe A, both the Horſes will be of equal value: But if the ſaid Trappings be laid on the other Horſe, he will be double the value of the firſt. How much did the ſaid Horſes coſt?

A = $33\frac{1}{3}$. B = 50. Trap^s = $16\frac{2}{3}$

41. A Vintner has two ſorts of Wine, *viz.* A and B: which if mix'd in equal Parts, a Flaggon of mix'd will coſt 15 Stivers; but if they be mix'd in a *ſeſqui-alter* proportion, as if you ſhould take 2 Flaggons of A as often as you take 3 of B, a Flaggon will coſt 14 Stivers. Requir'd the Price of each Wine ſingly. A = 20 B = 10

42. A Son ask'd his Father how Old he was; his Father anſwer'd him thus, if you take away 5 from my Years, and divide the remainder by 8, the Quotient will be $\frac{1}{3}$ of your Age: but if you add 2 to your Age, and multiply the whole by 3, and then ſubſtract 7 from the Product, you will have the number of the Years of my Age. What was the Age of the Father and the Son?

53) 18)

43. To find out two Numbers, to the Sum whereof if you add 6, the whole ſhall be double the greater; and if you ſubſtract 2 from their difference, the remainder will be half of the leaſt.

n° = 8 = 14

44. To find two Numbers, the Product whereof is 240, and the triple of the greater divided by the leſs is 5. n° = 20 & 12

45. Two Men have a mind to purchaſe a Houſe rated at 1200 Pounds; ſays A to B, if you give me $\frac{2}{3}$ of your Money, I can purchaſe the Houſe alone; but ſays B to A if you will give me $\frac{3}{4}$ of yours, I ſhall be able to purchaſe the Houſe. How much Money had each of them?

A =800

B =600

46. Some young Men and Maids had a reckoning of 37 Florins to pay for a Treat, and this was their Conditions, that every young Man ſhould pay 3 Florins, and every Maid 2. Now, if there had been as many young Men as there were Maids, obſerving the ſame conditions, the reckoning would have come to ^{5}4 Florins leſs than it did. How many young Men and Maids were there?

if 5 No = 8 maids, 7 men
if 14 florins No 8 $\frac{1}{4}$, 6 $\frac{5}{6}$ men

47. A General, who had fought a Battle, upon reviewing his Army, whoſe Foot was thrice the number of his Horſe, finds that before the Battle $\frac{1}{12}$—120 of his Foot had deſerted, and of his Horſe $\frac{1}{20}+120$, beſides $\frac{1}{4}$ of his whole Army were ſent into Garriſons (reckoning the Sick and Wounded), and $\frac{3}{8}$ of his Army remain'd; the reſt, who were wanting, being either ſlain or taken Priſoners; now, if you add 3000 to the number of the ſlain, the Sum will be equal to half the Foot he had at the beginning. What were the numbers of each?

Horse: 10000
Foot: 30000

48. To divide 100 twice into two parts, ſo that the major part of the firſt diviſion may be treble the minor part of the ſecond diviſion; and the *major* part of the ſecond may be double the *minor* part of the firſt. No a = 60. b = 40. d = 80. c = 20

49. To divide 30 twice into two parts, ſo that the *major* part of the firſt diviſion with the *minor* of the ſecond may be 33; and the Sum of the *minor* parts ſubſtracted from the Sum of the *major*, may leave 14 remaining. No = 20, 10) 17, 13,

50. A Man, his Wife, and his Son's Ages make up 96 Years, ſo that the Husband's and Son's Years together make the Wife's + 15; but the Wife's and the Son's make the Husband's + 2. What was the Age of each?

ages = 47. 40 $\frac{1}{2}$. 8. $\frac{1}{2}$

51. Three Merchants from three different Fairs meet together at an Inn, where they reckon up their Gains, and find them the Sum of 780 Florins. Moreover, if you add the Gain of the firſt and ſecond, and ſubſtract the Gain of the third from the Sum, there remains the Gain of the firſt, +82 Florins; but if you add the Gain of the ſecond and third, and from the Sum ſubſtract the Gain of the firſt, there remains the Gain of the third —43 Florins. What was the Gain of each?

a = 316, b = 273, c = 191

52. Three Perſons, A, B, C, owe a certain Sum of Mony, ſo that A and B together owe 210 Florins; B and C, 290, and C and A 400. What did each of them owe?

A = 160. B = 50. C = 240

53. To find three Numbers, ſo that the firſt and half the remainder, the ſecond and $\frac{1}{3}$ of the remainder, and the third and $\frac{1}{4}$ of the remainder, may always make 34. a = 10, e = 22, y = 26

4	9	2
3	5	7
8	1	6

54. Let a Square be divided into 9 ſmall Squares: we are to find and diſpoſe the Numbers through the ſeveral ſmall *Areas*, ſo that the Sum of every three, taken either laterally or diagonally, may be always 15.

55. [Theorem] Let any Numbers whatſoever be given, if you ſubſtract every leſs Number from that which is the next greateſt: I ſay that the Sum of thoſe differences is equal to the difference of the greateſt and leaſt Numbers.

720 5 — 15 — 25 — 35 — 5 = 30

10 + 10 + 10 = 30

OF

OF

Quadratic Equations,

And their Solutions.

THere are three Caſes or Forms of Quadratic Equations, as follows:

The Firſt CASE.

$$\underline{x^2 = ax + b^2}$$

Solution $\frac{1}{2}a$ mult.

$$\underline{\frac{1}{2}a}$$

$$\frac{1}{4}a^2 + b^2$$

$$x = \sqrt{\tfrac{1}{4}a^2 + b^2} + \tfrac{1}{2}a = BD$$

Proof or Reſtitution.

$$x - \tfrac{1}{2}a = \sqrt{\tfrac{1}{4}a^2 + b^2}$$

$$\underline{x - \tfrac{1}{2}a}$$

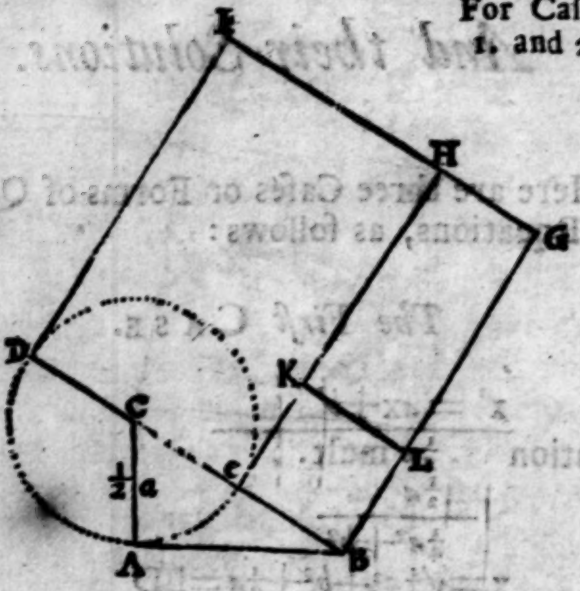
$$x^2 - ax + \tfrac{1}{4}a^2 = \tfrac{1}{4}a^2 + b^2$$

$$\therefore x^2 = ax + b^2$$

Demonſtration of the Firſt Caſe.

Let AC$=\frac{1}{2}a$, and AB$=b$: which being ſet at right Angles in A, upon the Center C, with the Radius CA, deſcribe the Circle AD*e*, and drawing the right line DCB$=x$, make upon it the Square BDFG$=x^2$; draw likewiſe *e*H parallel to the Line BG.

For Caſe, 1. and 2.



I ſay, 1.) That $x^2=ax+b^2$. For firſt DC*e*$=a$, and DF$=x$, make the Rectangle DH$=ax$; and then (by the 36. 3.) the Rectangle DB*e* [that is, GB*e*] = the Square AB, that is, b^2. Therefore $ax+b^2=x^2$.

I ſay, 2.) That $x=\sqrt{\frac{1}{4}a^2+b^2}+\frac{1}{2}a$. For the Square of AC$=\frac{1}{4}a^2$, and □ of AB$=b^2$: to which (by the 47. 1.) □ of BC is equal; therefore BC$=\sqrt{\frac{1}{4}a^2+b^2}$: add CD=CA$=\frac{1}{2}a$ and BD will=BC+CD. That is, $x=\sqrt{\frac{1}{4}a^2+b^2}+\frac{1}{2}a$. Q. E. D.

The

The Second CASE.

$$\frac{x^2 = -ax + b^2}{\text{Solution } -\tfrac{1}{2}a \text{ mult.}}$$

$$\frac{-\tfrac{1}{2}a}{+\tfrac{1}{4}a^2 + b^2}$$

$$x = \sqrt{\tfrac{1}{4}a^2 + b^2} - \tfrac{1}{2}a = eB.$$

The Proof or Reſtitution.

$$x + \tfrac{1}{2}a = \sqrt{a^2 + b^2}$$

$$\text{mult. } \frac{x + \tfrac{1}{2}a}{x^2 + ax + \tfrac{1}{4}a^2 = \tfrac{1}{4}a^2 + b^2}$$

$$\because x^2 \quad = -ax + b^2$$

Demonſtration of the Second Caſe.

The Figure being conſtructed as before, now let $eB = x$; whence $DB = a + x$; the Square eL deſcribed upon $eB = x^2$.

I ſay now 1.) That $x^2 = b^2 - ax$. For $GB = DB = a + x$, and $LB = eB = x$; therefore $GL = De = a$; which together with $KL = eB = x$, makes ▭ $GK = ax$. Therefore ▭ Ge [that is ▭ GK + □ of eL] $= x^2 + ax$]. Which very ▭ Ge, being ſhewn in the foregoing Caſe = □ of $AB = b^2$; $x^2 + ax$ will $= b^2$ and $x^2 = b^2 - ax$.

I ſay 2.) that $x=\sqrt{\frac{1}{4}a^2+b^2}-\frac{1}{2}a$. For $Ce=\frac{1}{2}a$, and $eB=x$, whence $CB=\frac{1}{2}a+x=\sqrt{\frac{1}{4}a^2+b^2}$ by the foregoing; and therefore $x=\sqrt{\frac{1}{4}a^2+b^2}-\frac{1}{2}a$. Q. E. D.

The Third CASE.

$x^2 = ax - b^2$

Solution $\frac{1}{2}a$ mult.

$\frac{1}{2}a$

$\frac{1}{4}a^2 - b^2$

$x=\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-b^2}$ The greater Root, $=BD$

and $x=\frac{1}{2}a-\sqrt{\frac{1}{4}a^2-b^2}$ The leſſer Root, $=BE$

For this third Caſe has two true Roots; of which ſometimes the greater, ſometimes the leſs, and often both will ſerve for the conſtruction of the Queſtion propos'd, as practiſe will teach us.

The Proof or Reſtitution.

$$
\begin{array}{lll}
\text{mult. } x-\frac{1}{2}a & = & \pm\sqrt{\frac{1}{4}a^2-b^2} \\
\quad x-\frac{1}{2}a & & \\
\hline
x^2-ax+\frac{1}{4}a^2 & = & \frac{1}{4}a^2-b^2 \\
x^2-ax & = & -b^2 \\
x^2 & = & ax-b^2
\end{array}
$$

Demon-

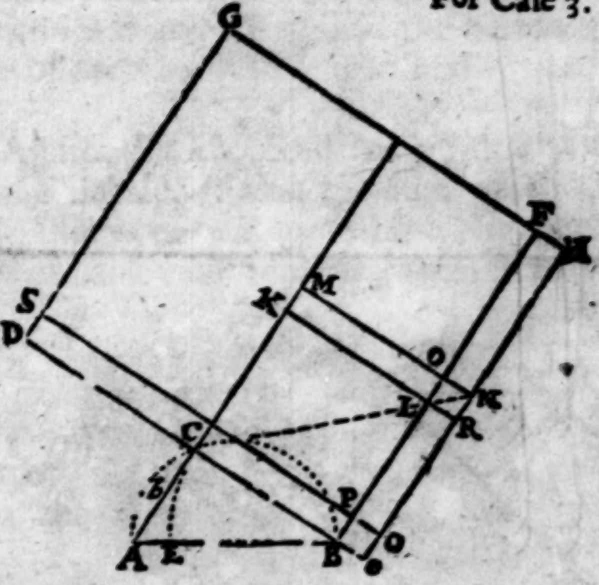
Demonſtration of the third Caſe.

In the following Figure, let AB$=\frac{1}{2}a$, and AC$=b$. upon AB deſcribe a Semicircle, lay off AC in the peripheria, and joyn BC: which being produc'd, make CD, and alſo C*e*, =AB.

Now (1.) let DB$=x$, the greater Root, upon which ■DF being made will $=x^2$. Then let *e*H be drawn parallel to the line BF, meeting GF produç'd in H, ▭DH will $=ax$.

I ſay now 1.) that $x=\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-b^2}$ by the 47. 1 and conſtruction.

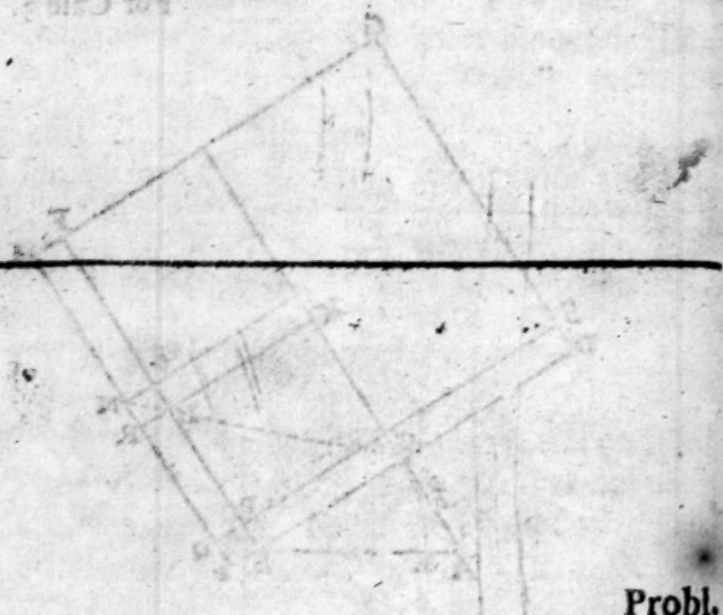
For Caſe 3.



I ſay 2.) that $x^2=ax-b^2$. For making the Square CN upon the line C*e*$=\frac{1}{2}a$, it will $=\frac{1}{4}a^2$: and alſo upon CB$=\sqrt{\frac{1}{4}a^2-b^2}$, deſcribing the

Square CL, it will $=\frac{1}{4}a^2-b^2$. Wherefore the gnomon KNB will $=b^2$: but ▭BH is equal to this gnomon, which is thus demonſtrated. Draw the Diameter CN, then the complements ML, *e*L are equal (by the 43. 1) but alſo ▭OH$=e$L: For if BO$=e$N$=e$C be taken from BF$=$BD, OF $=$BC$=$BL will remain: And ſince alſo ▭OH is $=$to OK, ▭BH will $=$ the gnomon KNB$=b^2$; and conſequently ■DF$=$▭DH $-$ ▭BH; that is, $x^2=ax-b^2$. Q. E. D.

But (2) let *Be*, $=$AE, $=x$, the leſs Root: upon which the Square BQ being made will $=x^2$, and ▭*e*S$=ax$: from which if you take away ■BQ, there will remain ▭BS$=$BH$=b^2$. Whence the reſt eaſily follows.



Probl. 56. To find a Number, which being multiply'd by 6, and the product ſubſtracted from the Square of the Number to be found, the remainder will be 280. Nº = 20

57. To find a Number which being multiply'd by 8, and the Product added to the Square of the Number to be found, the Sum will be 660.

Nº = 22

58. To divide 140 into two parts, ſo that the Product of thoſe parts may = the Square of 56, that is 3136. Nº = 112 × 28 = 3136

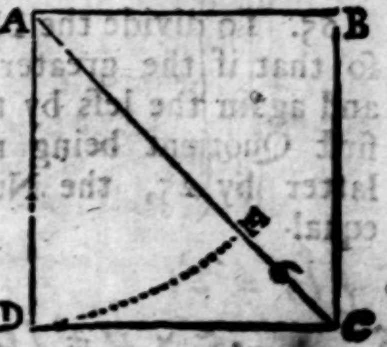
59. Let 969 Soldiers be drawn up into an oblong Battle, ſo that the difference of the greater and leſs ſides is 40. Requir'd the Number of the Soldiers of each rank in length and breadth?

Nº = 57 × 17 = 969

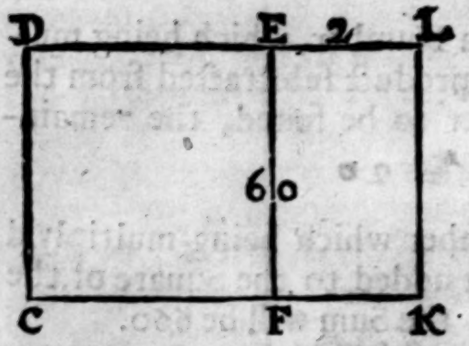
60. Again, let 480 Soldiers be drawn up into an oblong Battle, ſo that the Sum of the greater and leſs ſides is 52. Requir'd the Number of the Soldiers of each rank in length and breadth.

Nº 12 × 40 = 480

61. In the Square ABCD is given the difference of the Diagonal and the ſide, that is EC=6. Requir'd the ſide of the Square.



$\sqrt{72}\,:+6$



62. The Rectangle EK is added to the Square DF, [being of the ſame heighth;] whoſe breadth EL is given = 2, and alſo the *area* of the whole compound rectangle DK, =60. Requir'd the ſide of the Square.

6.8146 X 8.8146

63. A Man buys ſome Ells of Cloth for 70 Florins; and finds that if he had 4 Ells more, he had then bought every Ell, 2 Florins cheaper. How many Ells did he buy? 10 at 7 [illegible] Ell = 70

5:7::10:14 14 at 5 = 70. 5 = 7 - 2

64. A ſet of boon Companions dining at an Inn, the Reckoning in all came to 175 Shillings: but, before the Bill was paid off, two of them ſlunk away, and then the Club of thoſe that remain'd came to 10 Shillings a Man more. How many were there in Company? 7 : 5 : 35 : 25

first No = 7. & 4d 2d = 5

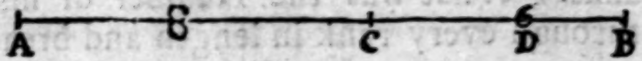
65. To divide the Number 21 into two parts, ſo that if the greater be divided by the leſs, and again the leſs by the greater; and then the firſt Quotient being multiply'd by 4, and the latter by 25, the Numbers produc'd may be equal.

$6 + 15 = 21.$

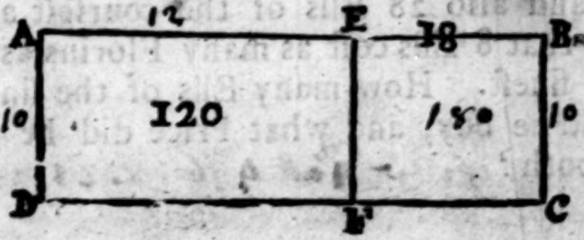
$\frac{15}{6} = 2.5 \times 4 = 10$

$\frac{6}{15} = 0.4 \times 25 = 10$

66. Let



66. Let the line AB be divided in C, ſo that AC may be 8, and CB 6: we are to divide the ſame line AB in D, ſo that the Rectangle under AD and DC may be equal to the Rectangle under AC and CB, or to the Product from 8 and 6, which is 48. Requir'd the Segment CD. = 4,



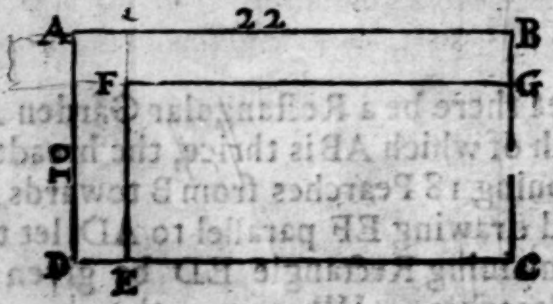
67. Let there be a Rectangular Garden ABCD, the length of which AB is thrice, the breadth AD: and reckoning 18 Pearches from B towards A, that is BE, and drawing EF parallel to AD, let the *area* of the remaining Rectangle ED be given = 120 ſquare Pearches. What was the length and breadth of the ſaid Garden? Length = 30 breadth = 10

68. Let 600 Soldiers be diſpos'd into an oblong Battle: which the Colonel willing to make broa-

der, finds that if he takes away 10 ranks from the length, he ſhall augment the breadth with two ranks. What was the Number of his Soldiers through every rank in length and breadth?
breadth = 10, Length = 60

69. A Man buys a Horſe, which he ſells again for 56 Crowns, and gains as many Crowns in 100 as the Horſe coſt him. How much did he give for the Horſe? 40

70. A certain Linnen Draper buys two ſorts of Linnen for 30 Florins, one finer, the other courſer: An Ell of the fineſt coſt as many Florins as he had Ells: and alſo 28 Ells of the courſeſt at ſuch a Price, that 8 Ells coſt as many Florins as one Ell of the fineſt. How many Ells of the fineſt Linnen did he buy, and what Price did he give for them both? finer $\overset{Ell}{4}$ att 4 florins. 28. at ½ a flor



71. In a certain rectangular Garden, the length of which AB is 22, and the breadth AD is 10, the walk DG is to be made, in a ſituation parallel to the ſides of the Figure, ſo that the *Area* of

the ſaid Walk or Gnomon DG may be equal to the remaining Rectangle FC [or that the Gnomon DG may be half of the whole Figure ABCD propos'd. Requir'd the breadth of the ſaid Gnomon DE, BG? $= 16 - \sqrt[2]{146}$. i.e, *fere* 4

72. Of three proportional Numbers there is the middle Term given =12, and the difference of the Extremes = 10. Requir'd the Extremes?

8 : 12 :: 12 : 18

73. Of three proportional Numbers is given the Sum of the firſt and ſecond =10, and the difference of the ſecond and third = 24. Requir'd the ſeveral Numbers?

2 : 8 :: 8 : 32

74. Of four proportional Numbers there is given the third =12, alſo the Sum of the firſt and ſecond =8; beſides the ſecond Number being ſubſtracted from its Square, the remainder is to be the fourth. Requir'd the ſaid Numbers?

3 : 5 : 12 : 20

75. Of four Numbers in continu'd proportion there is given the Sum of the means =24, and likewiſe the Sum of the Extremes = 56. Requir'd the ſaid Numbers [ſuppoſing that the firſt is the leaſt of all.] 2 : 6 :: 18 : 54

76. Two Country-women A and B, carry 100 Eggs together to Market, and in the ſale of them, one took as much Mony as the other: but

A

8 : a :: a : 18
aa = 144
a = 12
8 : 12 :: 2 : 3
40 : 60 :: 2 : 3

A [who had the largeſt, and conſequently the beſt Eggs] ſays to B, had I carry'd as many Eggs as you, I ſhould have had 18 Stivers for them; B replies, if I had brought as many Eggs as you, I ſhould have had but 8 Stivers for them. How many Eggs had each? 60 at 12^{d}, 40 at 12

77. Two Country-men A and B ſell their Corn at different Prices: A ſells 20 Buſhels; and B receiv'd for one Buſhel as many Florins as he ſold Buſhels: A perceives that if he had ſold as many Buſhels as B receiv'd Florins, he ſhould then have receiv'd 252 Florins; but both together receiv'd 176 Florins. How many Buſhels did B ſell, and what Price had A?
B. 6 Bush A att 7^{s} p Bushe

78. Two Merchants ſell 21 Ells of Cloth: The firſt ſells 1 Ell for as many Crowns, as is $\frac{1}{5}$ of the number of Ells that the ſecond had; and the ſecond ſells 1 Ell for as many Crowns, as is $\frac{1}{3}$ of the number of the Ells that the firſt had. The Sale being over, they had taken 48 Crowns in all. How many Ells did each ſell, and at what Price?
24 | 15 att 2^{e}, 6^{e} att 3 or 15 att 1$\frac{1}{5}$ & 6 at 5

79. Two Merchants have a parcel of Silk; the firſt 40 Ells, the ſecond 90: The firſt ſells for a Crown $\frac{1}{3}$ of an Ell more than the ſecond: When the Sale was over, they had taken between them 42 Crowns. How many Ells did each of them ſell for a Crown? first 3$\frac{1}{3}$ – 2^{d} 3 Ells

$\frac{40}{3\frac{1}{3}} = 12$

$\frac{90}{3} = \frac{30}{42}$

80. To find a Number, to the quadruple of which if you add 91, the whole ſhall be to the Square of the Number Sought, as 3 to 4 :: 147 : 196
114(

81. To find a Number, from the double of which if you ſubſtract 12, the Square of the remainder leſs 1, will be nine times the Number Sought. 22 − 12 [illegible] = 100 − 1 = 99. 11 × 9 = 99

82. To divide the Number 19 into two parts, ſo that the Sum of the Squares of the parts will be 193. N° 7 × 7 = 49, 12 × 12 = 144 + 49 = 193

83. To divide 7 into two parts, ſo that the difference of the Squares, which are made from the treble of the leſs part, and the double of the greater, may be 17. N° 3; 4, 81 − 64 = 17

84. A Man buys a Piece of Linnen, and by ſelling it again, he gains 12 Florins — $\frac{1}{10}$ of what he bought it for: And finds by this means that he had gain'd as much for 100 Florins, as the Linnen coſt him. What price was the Linnen bought and ſold at?. Bought att 80, & sold att 89

85. A Man buys 18 Ells of Cloth of different ſorts and colour, ſuppoſe red and black; what he bought of each coſt 40 Florins: And he pays for every Ell of red Cloth 1 Florin more than for the black. How many Ells of each ſort did he buy? = 10 + 8 = 18
[illegible] 4 5

86. A Man buys 120 Pounds of Pepper, and as many of Ginger: and receiv'd for a Florin one Pound of Ginger more than Pepper. So that the whole Price of the Pepper came to 6 Florins more than the Price of Ginger. How many Pounds of each did he buy for a Florin? 5 ℔ of
4 ℔ of pepper

87. A Man buys 80 Pounds of Pepper and 36 Pounds of Saffron, ſo that for 8 Florins he had 14 Pounds of Pepper more than he had of Saffron for 26 Florins, and what he laid out amounted to 188 Florins. How many Pound of Pepper had he for 8 Florins, and how many of Saffron for 26? = 6 ℔ of saforn & for 8 florins
20 ℔ of pepper

88. A and B between them owe 174 Florins. A pays 8 Florins a Day, and B pays the firſt Day 1 Florin, the ſecond 2, the third 3, and ſo on. In how many Days will they clear the Debt, and how much did each of them owe? A owes 96
B 78 paid in 12 days

89. A certain Man intends to Travel as many Days as he has Crowns: It happens that every following Day of his Journey he had as many Crowns as he had the Day before, beſides two Crowns over and above; and when he came to his Journey's end he finds he had in all 45 Crowns. How many had he at firſt?

5+7+9+11+13 = 45

90. A certain Traveller goes 9 Miles a Day, three Days after another follows him, who the firſt Day Travels 4 Miles, the ſecond 5, the third 6, and ſo on, gaining a Mile every Day. In what time will he overtake the former?

in. 14 Days & $\frac{2}{3}$ of a day

91. Two Travellers ſet out at the ſame time from two Cities, the one from A, and the other from B, which are 70 Miles diſtant from one another; one of them goes 6 Miles every Day; and the other, 2 Miles the firſt Day, $2\frac{1}{2}$ the ſecond, three the third, and ſo on, adding $\frac{1}{2}$ a Mile to every Day's Journey. In what time will they meet with one another? in 7 days & $\frac{7}{23}$ of a day
if it were 78 Miles distance it would be just 8 days

92. Again, Two Travellers ſet out at the ſame time from two Cities, the one from A, the other from B, which are 120 Miles diſtant from one another; the firſt goes 5 Miles a Day, and the other 3 Miles leſs than the Number of Days in which they meet. When will they meet? in 10 Days

93. A Poſt ſets out from A towards B, who Travels 8 Miles a Day: after he had gone 27 Miles, another ſets out from B to meet him, who goes every Day $\frac{1}{20}$ of the whole Journey or diſtance of the Places A and B, and meets the firſt Poſt after ſo many Days as is $\frac{1}{20}$ of the ſaid diſtance. Requir'd the diſtance of A and B. = 180 Miles

94. Two Merchants A and B go Partners, B brings 420 Florins, and A receives out of the gains 52 Florins, and the Sum of both their Shares of Stock & Gain is 854 Florins. How much did A bring, and how much did B receive out of the gains?

312 : 52 :: 420 : 70

95. A Son asks his Father how Old he was? his Father replies thus; If you take 4 from my Age, the remainder will be thrice the number of your Years: But if you take 1 from your Age, half the remainder will be the Square Root of my Age. Requir'd the Age of the Father and Son?

Sons : 15 · Fathers 49

96. To find two Numbers, the Sum of whoſe Squares may be 317, and the Product, if they be multiply'd by one another, 154.

11 & 14.

97. To find two Numbers, the Product of which may be 108, and the difference of the Squares 63.

12, & 9,

98. Two Farmers ſell two ſorts of Corn: A ſells 6 Buſhels; B receives in all for his 20 Florins: Now, ſays B to A, if we add the number of my Buſhels to the number of your Florins, the Sum will be 28: Says A to B, and if I add the Square of my Florins to the Square of your Buſhels, the Sum will be 424. How many Buſhels did B ſell, and how many Florins did A receive?

18 Bush & 10 Florins

99. To find two Numbers, the firſt of which +2, multiply'd into the ſecond —3, may produce 110: and on the contrary firſt —3, multiply'd by the ſecond +2, may produce 80.

ſuppose $a = 8$, & $b = 14$

[illegible]

Tables of Numerical Powers.

√	Qua	√	Qua.	√	Quad.	√	Quad.	√	Quadr.
1	1	21	441	41	1681	61	3721	81	6561
2	4	22	484	42	1764	62	3844	82	6724
3	9	23	529	43	1849	63	3969	83	6889
4	16	24	576	44	1935	64	4096	84	7056
5	25	25	625	45	2025	65	4225	85	7225
6	36	26	676	46	2116	66	4356	86	7396
7	49	27	729	47	2209	67	4489	87	7569
8	64	28	784	48	2304	68	4624	88	7744
9	81	29	841	49	2401	69	4761	89	7921
10	100	30	900	50	2500	70	4900	90	8100
11	121	31	961	51	2601	71	5041	91	8281
12	144	32	1024	52	2704	72	5184	92	8464
13	169	33	1089	53	2809	73	5329	93	8649
14	196	34	1156	54	2916	74	5476	94	8836
15	225	35	1225	55	3025	75	5625	95	9025
16	256	36	1296	56	3136	76	5776	96	9216
17	289	37	1369	57	3249	77	5929	97	9409
18	324	38	1444	58	3364	78	6084	98	9604
19	361	39	1521	59	3481	79	6241	99	9801
20	400	40	1600	60	3600	80	6400	100	10000
√c.	Cub.	√c.	Cub.	√c.	Cub.	v√	Q-qu.	v√	Q.qua.
1	1	11	1331	21	9261	1	1	11	14641
2	8	12	1728	22	10648	2	16	12	20736
3	27	13	2197	23	12167	3	81	13	28561
4	64	14	2744	24	13824	4	256	14	38416
5	125	15	3357	25	15625	5	625	15	50625
6	216	16	4096	26	17576	6	1296	16	65536
7	343	17	4913	27	19683	7	2401	17	83521
8	512	18	5832	28	21952	8	4096	18	104976
9	729	19	6859	29	24389	9	6561	19	130321
10	1000	20	8000	30	27000	10	10000	20	160000

Reductory

Reductory Table of Irrational Numbers.

√6 =	2√1½	√75 =	5√3	√200 =	5√8	√486 =	9√6
√8	2√2^2	√80	4√5	√216	6√6	√490	7√10
√10	2√2½	√90	3√10	√243	9√3	√500	10√5
√12	2√3^2	√96	4√6	√245	7√5	√512	8√8
√18	3√2	√98	7√2	√250	5√10	√539	7√11
√20	2√5	√99	3√11	√252	6√7	√567	9√7
√24	2√6	√108	6√3	√275	5√11	√600	10√6
√27	3√3	√112	4√7	√288	6√8	√640	8√10
√28	2√7	√125	5√5	√294	7√6	√648	9√8
√32 =	4√2	√128 =	8√2	√300 =	10√3	√700 =	10√7
√40	2√10	√147	7√3	√320	8√5	√704	8√11
√44	2√11	√150	5√6	√343	7√7	√800	10√8
√45	3√5	√160	4√10	√360	6√10	√810	9√10
√48	4√3	√162	9√2	√384	8√6	√891	9√11
√50	5√2	√175	5√7	√392	7√8	√1000	10√10
√54	3√6	√176	4√11	√396	6√11	√1100	10√11
√63	3√7	√180	6√5	√405	9√5	√1200	10√12
√72	3√8	√192	8√3	√448	8√7		
2√2 =	√8	2√5 =	√20	2√7 =	√28	2√10 =	√40
3√2	√18	3√5	√45	3√7	√63	3√10	√90
4√2	√32	4√5	√80	4√7	√112	4√10	√160
5√2	√50	5√5	√125	5√7	√175	5√10	√250
6√2	√72	6√5	√180	6√7	√252	6√10	√360
7√2	√98	7√5	√245	7√7	√343	7√10	√490
8√2	√128	8√5	√320	8√7	√448	8√10	√640
9√2	√162	9√5	√405	9√7	√567	9√10	√810
10√2	√200	10√5	√500	10√7	√700	10√10	√1000
2√3 =	√12	2√6 =	√24	2√8 =	√32	2√11 =	√44
3√3	√27	3√6	√54	3√8	√72	3√11	√99
4√3	√48	4√6	√96	4√8	√128	4√11	√176
5√3	√75	5√6	√150	5√8	√200	5√11	√275
6√3	√108	6√6	√216	6√8	√288	6√11	√396
7√3	√147	7√6	√294	7√8	√392	7√11	√539
8√3	√192	8√6	√384	8√8	√512	8√11	√704
9√3	√243	9√6	√486	9√8	√648	9√11	√891
10√3	√300	10√6	√600	10√8	√800	10√11	√1100
2√c.2 =	√c.16	2√c.3 =	√c.34	2√c.4 =	√c.32	2√c.5 =	√c.40
3√c.2	54	3√c.3	81	3√c.4	108	3√c.5	135
4√c.2	128	4√c.3	192	4√c.4	256	4√c.5	220
5√c.2	250	5√c.3	375	5√c.4	500	5√c.5	625
6√c.2	432	6√c 3	648	6√c.4	864	6√c.5	1080
7√c.2	686	7√c.3	1029	7√c.4	1372	7√c.5	1715
8√c.2	1024	8√c.3	1536	8√c.4	2048	8√c.5	2560
9√c.2	1458	9√c.3	2187	9√c.4	2910	9√c.5	3645
10√c.2	2000	10√c.3	3000	10√c.4	4000	10√c.5	5000

LINEAR ALGEBRA

In Order to the

Geometrick Conſtruction

OF

EQUATIONS.

PROPOSITION I.

To add a right Line given to a right Line given, and find their Sum.

SUppoſe a and b are to be added; the Sum of them will be $a+b=d$.

PROP. II.

From a greater right Line given to ſubſtract a leſs, and find their difference.

SUbſtract c from b; the difference will be $b-c=f$.

PROP.

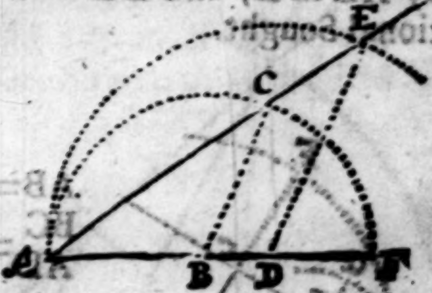
PROP. III.

To find a third proportional to two right Lines given [and that with the utmoſt exactneſs that is poſſible.]

Caſe 1.

If the difference of the given Lines be neither very ſmall, nor very great.

FRom the center B with the diſtance of the firſt Line AB, deſcribe the Semicircle ACF, in which place the ſecond Line AC from A, and produce it beyond C infinitely: then place the ſecond Line from A upon the Diameter of the Semicircle, ſuppoſe in D, and from D, with the diſtance DA, an Arch being deſcrib'd will determin the point E, in AC produc'd, and AE will be the third proportional Sought.

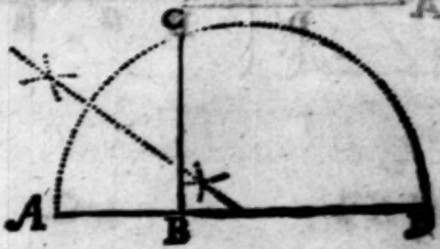


AB $= a$, firſt,

$\begin{matrix}AC\\AD\end{matrix} = b$, ſecond,

AE $= \frac{b^2}{a}$ third Line Sought.

Or thus.



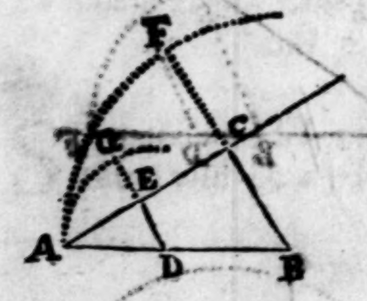
AB, firſt,
BC, ſecond,
BD, third ſought.

But as concerning this firſt Caſe, *viz.* where the difference of the given Lines is not immoderately great, ſeveral commodious Expedients may be made uſe of (as in the fourth Propoſition following) but not ſo in the other Caſes.

Caſe 2.

If the difference of the given Lines be vaſtly great, and the firſt is greater than the ſecond.

With the diſtance of the firſt Line AB, from the point B make an Angle of 60 Degrees ABF; from whoſe other leg BF cut off the ſecond Line BC, and draw AC. Now make AD= the ſecond line BC, and from the point D, with the diſtance DA (or any other) deſcribe another Angle of 60 deg. ADG, whoſe other leg DG will be cut by AC in E, and DE will be the third proportional Sought.



AB $=a$, firſt,

BC
AD $=b$, ſecond,

DE $=\frac{b^2}{a}$ third Sought.

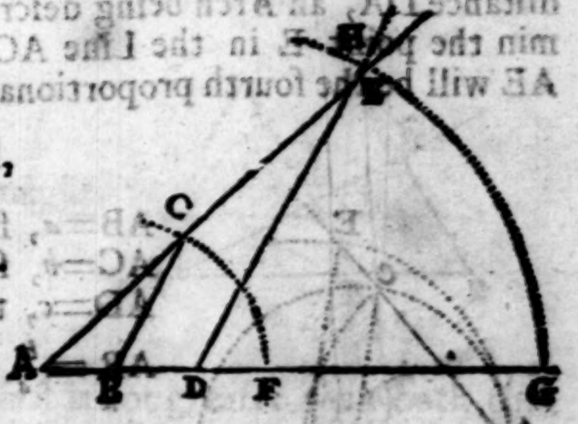
Caſe 5.

If the difference of the given Lines be vaſtly great, and the firſt be leſs than the ſecond.

1. From B, the end of the firſt Line, with the diſtance of the ſecond BC, make an Angle of 60 deg. BFC, and from the point A draw a right Line infinitely thro' C. 2. From D the end of the ſecond Line, with a ſufficient diſtance make an other Angle of 60 deg. GDH, and a right Line being drawn till it meet the infinite Line AC in E, DE will be the third proportional Sought.

For what elſe is to be obſerv'd as to this Propoſition, ſee the next.

AB=a, firſt,
BC
AD =b, ſecond,
DE=$\frac{b^2}{a}$ third Sought.



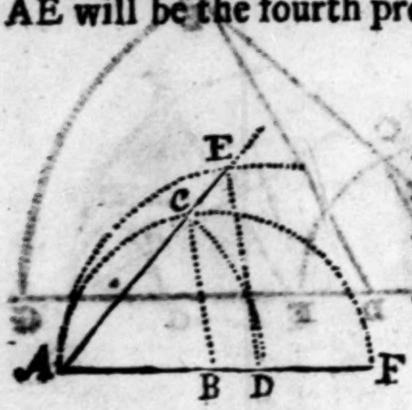
PROP. IV.

Three right Lines being given, to find a fourth proportional.

Caſe 1.

If of the Lines given (ſuppoſe the firſt and either of the reſt) the difference is not extraordinary great, and eſpecially if the firſt is leſs than either of the reſt.

With the diſtance of the firſt Line AB from the point B deſcribe the Semicircle ACF, in which place the ſecond AC from the point A, and produce it infinitely: then place the third from the point A upon the Diameter of the Semicircle, ſuppoſe in D; and from D, with the diſtance DA, an Arch being deſcrib'd will determin the point E in the Line AC produc'd, and AE will be the fourth proportional Sought.



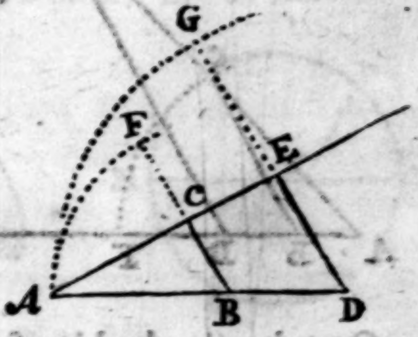
$AB=a$, firſt,
$AC=b$, ſecond,
$AD=c$, third,
$AE=\frac{bc}{a}$ fourth Sought.

Caſe 2.

If of the given Lines (ſuppoſe the firſt or either of the reſt) the difference is extraordinary great; and the firſt be greater than either of the reſt.

With the diſtance of the firſt Line AB, from the point B make an Angle of 60 deg. ABF, from whoſe other leg BF cut off the ſecond BC, and draw an infinite right Line from the point A thro' C: Now place the third Line from the point A in D; and from the point D, with the diſtance DA (or any other) make another Angle of 60 deg. ADG, whoſe other leg DG will cut the Line AC produc'd, and DE will be the fourth proportional Sought.

AB=a, firſt,
BC=b, ſecond,
AD=c, third,
DE=$\frac{bc}{a}$ fourth Sought.

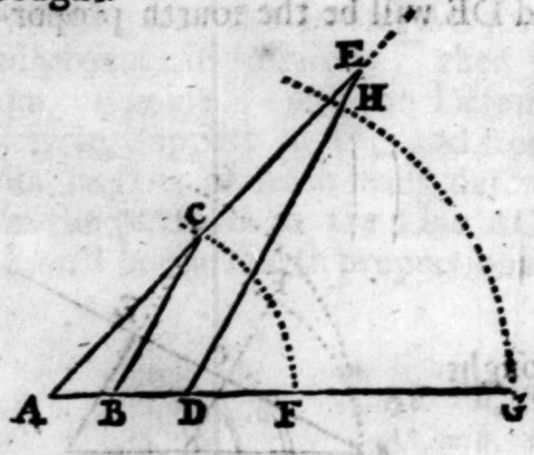


[*Note*, If the firſt is not greater than both the reſt, but then either of them only, as in this place the operation will proceed commodiouſly according to Caſe 1. *viz.* the ſecond and third being chang'd alternately.]

Caſe 3.

If the difference of the given Lines [*of the firſt and either of the reſt*] *be extraordinary great, and the firſt be the leaſt of the two.*

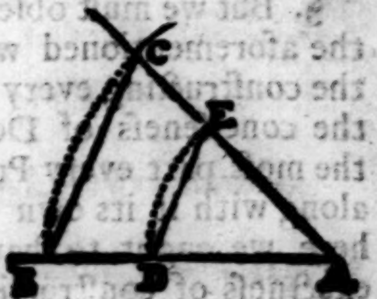
1. From B, the end of the firſt Line, with the diſtance of the ſecond BC, make an Angle of 60 deg. BFC; and draw an infinite right Line from the point A through C. 2. From D the end of the third, with a ſufficient diſtance, make another Angle of 60 deg. GDH, and having drawn the right Line DH till it meet the infinite Line AC in E, DE will be the fourth proportional Sought.



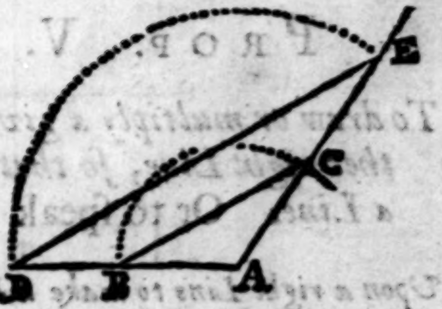
AB=a, firſt,
BC=b, ſecond,
AD=c, third,
DE=$\frac{bc}{a}$ fourth Sought

Or take the double of the firſt Line, and there will come forth half the Line Sought; and then the operation will be often reducible to *Caſe* 1.

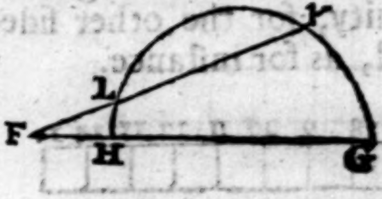
§. But unleſs the firſt of the given Lines be very ſhort, generally the Practiſe (of this and the foregoing Propoſitions) by the help of an equicrural Triangle, is accurate and expeditious.

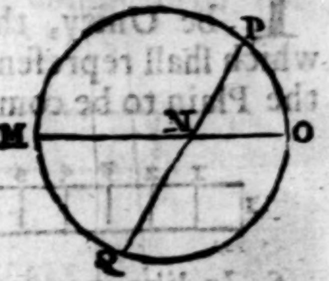


AB firſt,
BC ſecond,
AD third,
DE fourth Sought:



§. In the mean time the following ways may be noted from the 35 and 36 of the III. of *Euclid*.





FK firſt,
FG ſecond,
FH third,
FL fourth,

NQ firſt,
MN ſecond,
NO third,
NP fourth Sought:

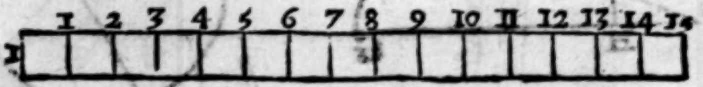
§. But we muſt obſerve, that not every one of the aforementioned ways are equally proper for the conſtructing every Problem, as far as concerns the conciſeneſs of Demonſtration, but that for the moſt part every Problem requires and carries along with it its own peculiar conſtruction; but here we ought to have a regard at leaſt to the exactneſs of conſtruction.

PROP. V. [Lemma.]

To draw or multiply a given right Line into another right Line, ſo that the product may be as a Line. Or to ſpeak Geometrically.

Upon a right Line to make a Rectangle; which (in its Numerical Value) *may be equivalent to the right Line propos'd.*

THis will be done if the multiplying right Line be Unity, that is, we aſſume a right Line which ſhall repreſent Unity, for the other ſide of the Plain to be compos'd, as for inſtance.



§. In like manner may a quantity of never ſo many Dimenſions be compos'd, which notwithſtanding ſhall repreſent only one Dimenſion.

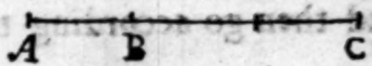
So likewiſe any quantity may be multiply'd into a Line; ſo that the Product may be equivalent to a quantity given.

PROP.

PROP. VI.

To multiply a given right Line by a number.

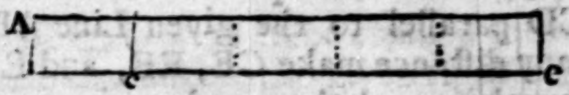
AS if the given right Line AB were to be multiply'd by 3, there will come forth the right Line AG.



PROP. VII.

To multiply a given Plain by a Number.

AS if the given Plai[redacted]ply'd by 5, there will come forth the Plain Ac.



PROP. VIII.

To divide a given Plane by a Line.

[Which is to reſolve a Fraction into a proportion.]

SUppoſe *ab* ▭ were to be divided by *c*, it will be as *c* the diviſor, is to *a*: ſo is *b* to $\frac{ab}{c}$, or to the Quotient Sought $= x$.

Note, If the plain to be divided be compound as $\frac{a^2+b^2}{c}$, $\frac{c^2+b^2-ab}{2a}$, &c.

Prop. IX.

To divide a given right Line by a right Line.

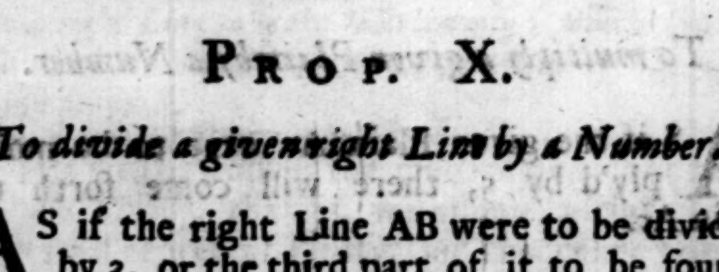
IMagin a Plain to be made upon the given right Line, whoſe other ſide is Unity [by 5 foregoing] and then go according to the 8 foregoing.

As if you was to divide b by a; let it be as a to b; ſo Unity $= e$ to $\frac{eb}{a} = \frac{b}{a}$.

Prop. X.

To divide a given right Line by a Number.

AS if the right Line AB were to be divided by 3, or the third part of it to be found. Draw CD parallel to the given Line AB; and then at any diſtance make CE, EE, and EF equal to one another, and the number given. Laſtly, joyn CA and FB till they meet at G. After that joyn GE, EE, *&c.* and you will have the Line divided as was requir'd.

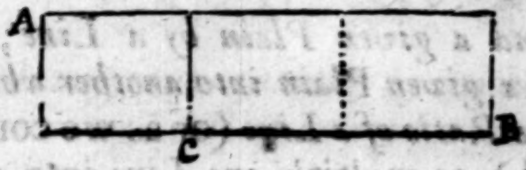


PROP.

Prop. XI.

To divide a given Plain by a number.

Suppoſe the propos'd Plain AB is to be divided by 3, the Quotient will be the third part of the Plain propos'd, ſuppoſe AC.



Prop. XII.

To change a given Plane into another, which may have a given ſide; or *to reduce a given Plain to a given Length or Breadth.*

Suppoſe ▭ bc were to be chang'd into another, whoſe one ſide = the given right Line d; or ▭ bc were to be reduc'd to the length d. Requir'd the breadth.

Divide (by the 8 foregoing) the propos'd ▭ bc by the given ſide d, and there will come forth the other ſide Sought; that is, let it be as d to b, ſo c to a fourth $= f$, = the ſide Sought, whence $df=bc$.

§. Hence it appears, after what manner any quantity of what dimenſions ſoever may be reduc'd to given Letters, or to a given Magnitude of ſides.

Suppoſe the quantity acb^2 were to be transform'd, ſo that the Letter a ſhall be found thrice in it, that is, the Cube of a. Wherefore firſt taking the Square b^2, it will be as a to b, ſo b to a third $=d$; whence $ad=b^2$, and $acad$ or $aacd=acb^2$. Then further taking the Rectangle cd, it will be as a to c, ſo d to a fourth $=f$; whence $af=cd$, and $aaaf$ or $a^3f=acb^2$, which was the thing deſir'd.

PROP. XIII.

To expound a given Plain by a Line; or *to change a given Plain into another which ſhall have the Ratio of a Line* (or as we commonly ſpeak) *to multiply one Line into another, according to any Line aſſum'd for Vnity.*

Divide the propos'd Plain by Unity, the Quotient will give the thing Sought. Suppoſe the Square ab were to be expounded by a Line, it will be as Unity, $=e$, to a, ſo b to a fourth $=f$; whence $ef=ab$, and alſo $ef=$ the Line it ſelf f, by the 8th foregoing.

§. Hence it appears after what manner any quantity of what Dimenſions ſoever may be expounded by a Line, or reduc'd to a linear Value. For Example, the quantity a^2b^2 is to be expounded into a Line.

Suppoſe Unity $=e$, and firſt having aſſum'd the Square a^2, it will be as e to a, ſo a to a third $=d$, whence $ed=a^2$. In like manner having aſſum'd the Square b^2, it will be, as e to b, ſo b to a third $=f$, whence $ef=b^2$, and $eedf=a^2b^2$: therefore at laſt having aſſum'd the Rectangle df it will be, as e to d, ſo f to a fourth $=g$, whence $eg=df$;

and conſequently $e3g=a^2b^2$; and at the ſame time $e3g=g$ numerically; which was the thing deſir'd.

§. But if in the Example propos'd [a^2b^2] the quantity a itſelf be taken for Unity, we ſhall ſooner compaſs our deſign, that is, by reducing the Square b^2 only as follows: as a to b, ſo b to a third $=k$; whence $ak=b^2$; and further $a3k=a^2b2=k$.

Prop. XIV.

To divide any given quantity by another, [or *to reſolve any Fraction into a proportion.*]

Caſe 1.

When the Diviſor has fewer Dimenſions than the Dividend. Which is ordinary Diviſion, as

$$\frac{bd^2}{ac},\ \frac{a^2df}{bg},\ \frac{a^2b^2+bf^3}{abf-af^2}.$$

Caſe 2.

When the Diviſor and Dividend have equal Dimenſions, as $\frac{ab}{c^2}$, $\frac{d^2f}{ab^2}$, $\frac{b^2ac}{f^2d^2}$.

Then in the dividend you muſt take in Unity by the 5 foregoing, which (for Example) may $=e$; therefore it will be $\frac{eab}{c^2}=\frac{ab}{c^2}$ propos'd; and ſo in the reſt.

Caſe 3.

When the Diviſor has more Dimenſions than the Dividend, as $\frac{b}{a^2}$, $\frac{f^2}{cd^2}$, $\frac{ac^2}{b^2f^3}$.

Then in the Dividend you muſt take in Unity, as often as there is occaſion, ſuppoſe till the Dividend exceeds the Diviſor in number of Dimenſions. Wherefore Unity being ſuppos'd $=e$ as before, there will be plac'd

$$\frac{e^2b}{a^2}=\frac{b}{a^2},\ \frac{e^2f^2}{cd^2}=\frac{f^2}{cd^2},\ \&c.$$

Caſe 4.

When the Diviſor has fewer Dimenſions than the nature or kind of the Quotient require.

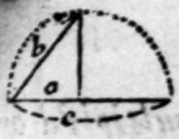
As if you propoſe $\frac{ab^2}{c}$, and the Quotient ought to be a Line. Then in the Diviſor you muſt take in Unity, that it may be $\frac{ab^2}{ec}=\frac{ab^2}{c}$, &c.

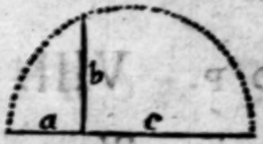
PROP. XV.

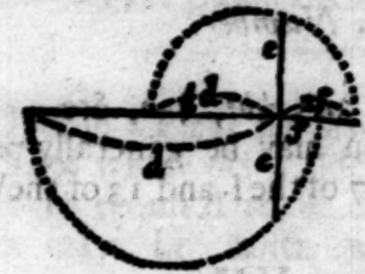
Two right Lines being given to find a mean Proportional.

LET a and c be given; the mean $=b$, will be found by the common Conſtruction.

And







And if either of the given Lines be too ſmall, take its double or treble, and the half or third part of the other, and ſo the Line Sought will be diſcover'd, as before.

For Example, Let *d* and *f* be given, the *mean* will be *e*, which is found both ways, as the Figure ſhews.

PROP. XVI.

To find the ſide of a ſingle *Plane given.*

A Mean proportional between the length and breadth of the propos'd Plan [ſuppoſe of a Rectangle] gives the Side Sought.

PROP. XVII.

To find the ſide of a given Plane which is increas'd or diminiſh'd by a Plane.

AS if there be given a^2+b^2, a^2-b^2, $a^2+b^2-c^2$, &c. The Conſtruction is manifeſt from the 47 of the I. and 31 of the III.

PROP. VIII.

To find the ſide of a given Plane, multiply'd or divided by a Number.

AS if there be given $3a^2$, $\frac{1}{2}b^2$, $\frac{2}{3}ab$, &c. The Conſtruction may be generally perform'd two ways, by the 47 of the I. and 13 of the VI.

PROP. XIX.

To find the ſide of a given Plane, affected as well with another Plane as Number (according to the 17, and 18 foregoing Propoſitions.)

AS if there be given $2a^3+\frac{1}{3}b^2$, $5b2+d^2-\frac{1}{2}c^2$, &c. The Conſtruction appears from what went before.

PROP. XX.

To find the ſide of any given Quantity which riſes to two Dimenſions.

AS if there be given $\frac{ab^2}{c}$ $\frac{a^2d^2}{bc}$ $\frac{c^2f^2+d^4}{c^2-d^2}$ &c. Let *b* be to *q*, as *c* to *a*; then a mean proportional between *b* and *q* will be the ſide Sought. And ſo of the reſt.

Prop. XXI.

To extract the Square Root from a given right Line.

SUppoſe a Plane be made upon the given right Line whoſe other ſide is Unity (which ſometimes is requir'd to be determinate, and ſometimes may be taken at liberty) and then (according to the XV. and XVI.) a mean proportional between the ſaid Unity, and the right Line propos'd, will give the Root Sought.

But if the given Unity be too ſmall, take its double or treble, and on the contrary the half or third part of the Line propos'd, as was hinted at the XV. Prop.

Prop. XXII.

From a given Quantity of three Dimenſions to extract the Square Root.

1. SUppoſe $\surd a^3$ were to be found out, which, as it is nothing elſe but the Square from $\surd a^2$ (that is a) and $\surd a$, or $\surd a^3$; is the ſame as $a\surd a$; therefore let $\surd a$ be found by XXI. foregoing, which being multiply'd into a will make a Plane; by the XIII. being reduc'd to the *Ratio* of a Line, you will have the Root Sought.

Or let the given Quantity a^3 be firſt expounded by a Line, and then make the extraction according to XXI.

But we muſt *Note*, that for this kind of Extraction (as alſo in other Caſes of this Propoſition) Unity cannot be aſſum'd at Liberty, but depends on the Quantity a.

2. Suppoſe $\sqrt{ab^2}$ were to be found out, which = the Square from b and $\sqrt{a}$, which being expounded by a Line you will have the Root ſought.

3. Suppoſe $\sqrt{bcd}$ were to be found, which = the Square from $\sqrt{b}$ and $\sqrt{cd}$. Therefore by the XVI. $\sqrt{cd}$ is found, and by the XXI. $\sqrt{b}$; and let the Square made from thence be expounded by a Line by the XIII. which will be $\surd$ Sought.

§. For exemplifying this by Numbers, let $a=4$; $b=3$; $c=2$; $d=5$.

PROP. XXIIII.

From a given Quantity of four Dimenſions to extract the Square Root.

Note, IN this place likewiſe Unity is determin'd by the given Quantities, nor can be aſſum'd at Liberty.

1. Suppoſe $\sqrt{a^4}$ were to be found out, which $=a^2$; and therefore let a^2 (by the XII.) be expounded by Line, and you will have the Root Sought.

2. Suppoſe $\sqrt{a^2b^2}$ were to be found out, which $=ab$, and therefore do as before.

3. Suppoſe $\surd b^2dc$ were to be found out, which $=b\surd dc$, inquire $\surd dc$ (by the XVI.) which with b will form a Plane, which Plane expounded by a Line (by the XIII.) will give the Root Sought.

4. Suppoſe $\surd abcd$ were to be found out; inquire $\surd ab$, and likewiſe $\surd cd$ (by the XVI.) and let the Plane made from both be reduc'd to a Line, (by the XIII.) and you will have the Root Sought.

PROP. XXIV.

From a given Quantity of four Dimenſions to extract the Bi-Quadratic Root.

The Firſt way by the help of Unity.

WHich in this place may be aſſum'd greater or leſs at liberty, for the true Root will always come forth at laſt; and conſequently any Letter of that given Quantity may be taken for Unity, which will make the Operation more compendious.

A general Rule for all Caſes.

Firſt let the Quadr. √ of the propos'd Quantity be found out (by the XXIII.); and then again extract the Root from that Root (by the XXI.) which will be √ Sought.

The Second way without Unity.

1. For $\sqrt{\sqrt{a^4}}$ there is no need of operation, for it $= a$.

2. For $\sqrt{\sqrt{a^2b^2}} = \sqrt{ab}$ a mean proportional between a and b will give the Root Sought.

3. For $\sqrt{\sqrt{b^2dc}} = b\sqrt{dc}$ a mean proportional between b and $\sqrt{dc}$ will give the Root Sought.

4. For $\sqrt{\sqrt{abcd}} = \sqrt{\sqrt{ab}\sqrt{cd}}$, a mean proportional between $\sqrt{ab}$ and $\sqrt{cd}$ will give the Root Sought.

PROP. XXV.

From a given compound quantity of four Dimenſions to extract the Bi-quadratick Root.

HEre likewiſe as in the foregoing Propoſition, any Unity may be aſſum'd at liberty; but yet that Letter of the Quantity propos'd, which occurs oftneſt in it, will moſt commodiouſly be taken for Unity.

1. For $\sqrt{\sqrt{b^2a^2+b^4}}$. Aſſuming b for Unity, let a^2 be expounded in a Line (by the XIII.) which may be called f. Likewiſe let b^2 be expounded by a Line, which will $= b$; whence $bf = b^2a^2$, and $\sqrt{bf} = \sqrt{b^2a^2}$; and alſo $\sqrt{bf} = a$, to which (by the XVII) the Root $b^4 = b^2 = b$ being added (ſince b is ſuppos'd $= 1$) you will have firſt $\sqrt{b^2a^2+b^4}$. From which if (by the XXI.) you again extract the Root you will have $\sqrt{\sqrt{b^2a^2+b^4}}$ Sought.

Hence this compendium of the Conſtruction: From $\sqrt{a^2+b^2}$ the Root extracted by the help of $b=1$ will give w Sought.

2. For $w\overline{a^4+b^4}.$ Suppoſing $a=1$, &c.

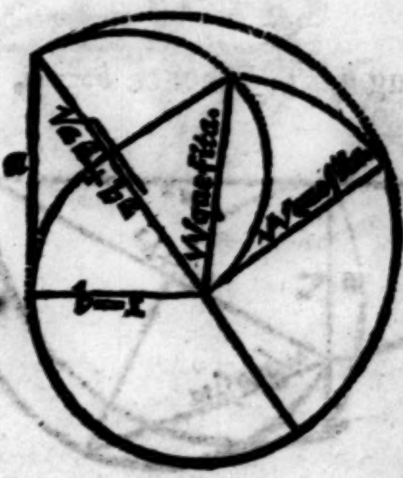
3. For $w\overline{bcd^2+bdc^2}.$ Suppoſing $d=1$, &c.

The general Rule of this Propoſition.

Let the Planes of the propos'd Quantities be expounded by Lines, (by the XIII.) with which proceed according to the XVI. XVII. That is to ſay, for the Square Root; and according to the XXI. for the Bi-quad. Root.

Here follow the Conſtructions.

§. For $w\overline{b^2a^2+b^4}.$ Suppoſing $b=1$.

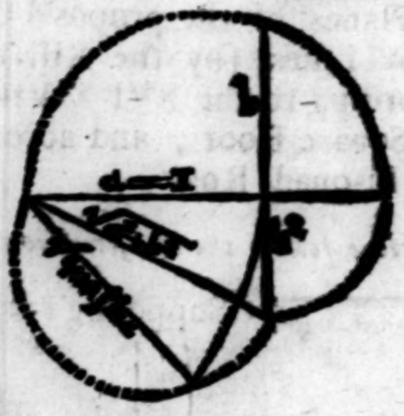


For quæſita read Sought.

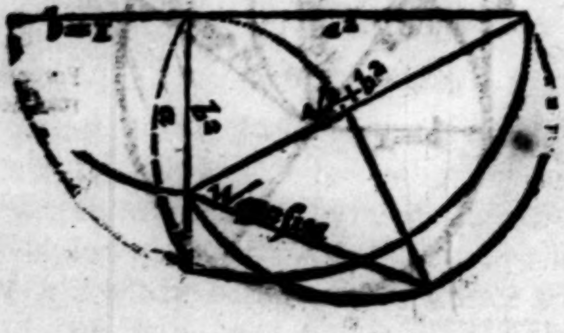
Farther the ſame will be had if you ſuppoſe $a=1$, but by a leſs compendious Operation.

§. For $w\sqrt{a^4+b^4}$. 1. Suppoſing $a=1$.

It will be $a^2=a$, now for b^2, as $a=1$ is to b, ſo is b to $\frac{b^2}{1}$ whence is gotten $\sqrt{a^2+b^2}$; from which the Root again being extracted by the help of $a=1$, will give the Root Sought.



2. Suppoſing $b=1$, whence $b2=b$, &c.

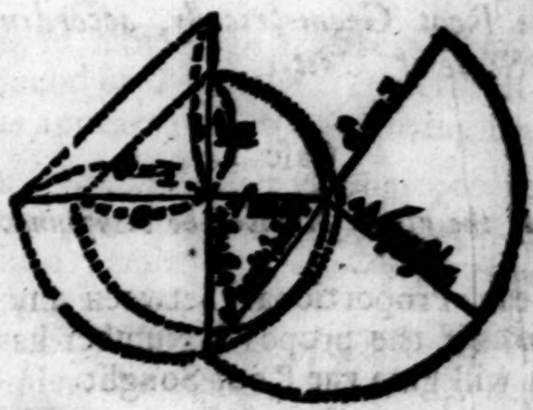


§. For $\surd\surd\,\overline{bcd^2+bdc^2}$. Suppoſing $d=1$.

1. For bc, as $d=1$ is to db, ſo is c to $\frac{bc}{1}=bc$.
2. $d^2=d$.
3. Mean Proport. between bc and $d^2=d$, $=$ $\sqrt{bcd^2}$.
4. bc as before.
5. $dc=c$.
6. Mean Proport. between bc and $c=dc=\sqrt{bdc^2}$.

Whence 7. is had $\sqrt{bcd^2+bdc^2}$.

And 8. by the help of $d=1$, $\surd\surd$ Sought. Which was the thing propos'd.



AN APPENDIX FOR *Irrational Numbers.*

PROP. XXVI.

From any given Number whatever to extract the Square Root Geometrically, according to any given Scale of Unities.

Caſe 1.

If the given Number be compound.

A Mean Proportional between any two *Factors* (if the propos'd Number has ſeveral Diviſors) will give the Root Sought.

EXAMPLES.

1. For $\surd 21$. inquir'd a mean Proport. between 3 and 7.
2. For $\surd 30$ inquir'd a mean Proport. between 2 and 15, or 3 and 10, or laſtly 5 and 6.
3. For $\surd 99$. inquir'd a mean Proport. between 3 and 33, or which is better, between 9 and 11.

Caſe

Caſe 2.

If the given Number be a Prime.

Divide it into two or more Parts, as the thing ſhall require, ſo that thoſe Parts, as far as may be, may compoſe rational Squares, or at leaſt Rectangles, and then proceed according to 47. I. *&c.*

EXAMPLES.

For √ 11	divide 11	into	4. 4. 3, or 9 and 2
13	13		9 and 4
29	29		25 and 4
37	37		25 and 12
101	101		100 and 1 or 81, 16, 4.

Further in moſt large Numbers, whether they be Compound or *Prime*, both the Operations may be us'd, as appears from the following Examples.

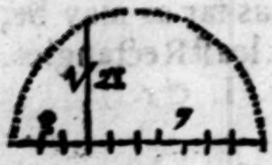
1. For √700. Subtr. 700 from 900 rem. 200; whoſe Root from *Caſe* 1. being Sought, and Subſtracted in a Semicircle from √900=30, you will have what was requir'd.

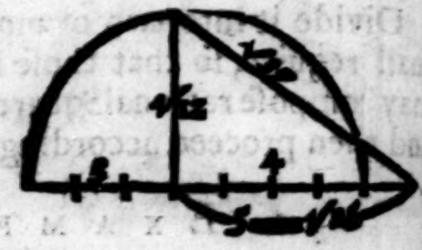
2. √6000. A mean Proport. between 60 and 100 will give the Root Sought: Or divide it as you may ſee at the following Figure of the Conſtruction.

3. For √4567. Diſtribution 1936 √ 44
2401 √ 49
4337 ſubtr.
4567
230 23
10

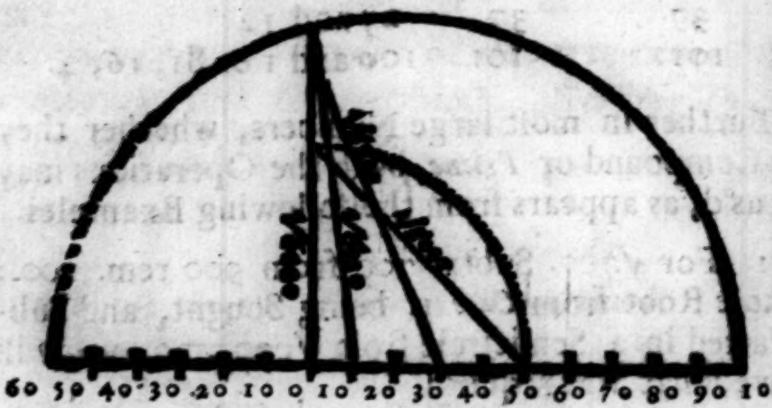
Here follow the Conſtructions.

For √21. For √37.





For $\sqrt{}\begin{array}{r|l} 6000 & 100 \\ 6\quad 0 & \end{array}$



Or thus: $$\begin{array}{r} 6000 \\ \hline 50 \\ 50 \\ \hline 2500 \\ 2 \\ \hline \end{array}$$ $$\left.\begin{array}{r} 5000 \\ \hline 30 \\ 30 \\ \hline 900 \\ \hline 10 \\ 10 \\ \hline 100 \end{array}\right) \text{add}$$

PROP.

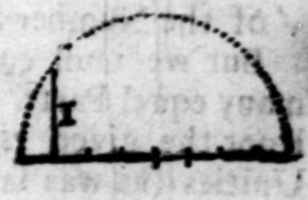
PROP. XXVII.

A right Line being given = to the Root of any given Number, to find a right Line = a proper Unity, or another rational Number.

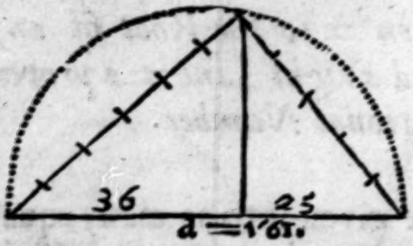
CUT the right Line given into as many equal Parts as the given Number contains Unities: and then find a *mean Proport.* between the whole Line, and one part of it; or between half and two parts, *&c.* which will be the Unity Sought.

Or, if the given Number be great, cut the Line in prorortion according to that Number, in ſuch a manner as that one part may = ſome Square Number: then find a mean Proportional between the whole Line and the ſaid Square Part, which will be √ of the Square Number beforemention'd; therefore this *mean* being cut into as many equal Parts, as the Root of the ſaid Number directs, you will have the thing Sought.

1. Let there be given a right Line $a = \sqrt{7}$.



2. Let there be given a right Line $d=\sqrt{61}$.



Let *d* be cut in the Proportion of 36 to 25, both of which are Square Numbers, and then find a mean Proportion between the whole and the greater Part, there will come out $\sqrt{36}=6$; or between the whole and the leſs part, there will come out $\sqrt{25}=5$.

PROP. XXVIII.

A right Line being given = the Root of any given Number, to find a right Line = the Root of any other Number propos'd, and that without the help of Unity.

A Mean Proportional between the whole Line and between as many parts of that Line as the Number of the Root Sought, will be √ of the Number Sought.

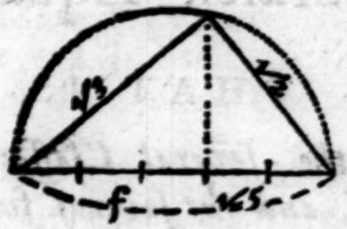
But we muſt cut the right Line given into as many equal Parts, as the Number given, (whoſe Root the given right Line repreſents) contains Unities (as was ſaid in the foregoing Prop.) Or rather, it is to be cut in ſuch a manner that one part ſhould = the other Number Propos'd, *viz.* whoſe Root is deſir'd.

Caſe

Caſe 1.

Where the Line or √ Sought is leſs than that Given.

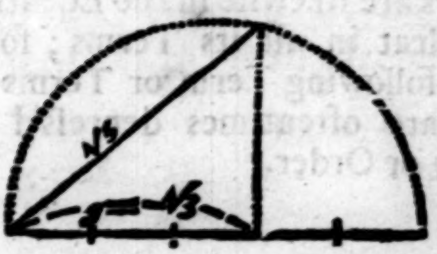
Let there be given the right Line $f=\sqrt{5}$; find out another $=\sqrt{3}$.



Caſe 2.

Where the Line or √ Sought is greater than that Given.

Let there be given interchangeably $\sqrt{3}=g$; find out the $\sqrt{5}$.



THE

Diſtribution and Solution of Algebraick Equations.

THAT IS,

Concerning the ſeveral Claſſes or Orders of Equations, and the ſeveral formulæ of thoſe Claſſes, together with their Geometrical Reſo-ſolution.

THere are as many Claſſes of *Algebraic Equations*, as there are degrees of Powers or Dimenſions in a progreſſional Scale of Quantities. And every Equation is ſaid to be of as many Dimenſions, as are found in the higheſt Power of the unknown Quantity; that is, if all the inferiour Powers are likewiſe in the Equation, ſo that it be compleat in all its Terms; for otherwiſe when ſome following Term or Terms are abſent, Equations are oftentimes depreſs'd to a more ſimple Claſs or Order.

CLASS

CLASS I.

Of Simple Equations or thoſe of one Dimenſion.

AND of theſe there are two Caſes:

1. Where x is equal to ſome Plane divided by a Line, or whatever is equivalent to it.
2. Where x is equal to the Root of any Plane or $\surd\surd$ of any Plane-Plana.

Of the firſt Caſe: [Which relates to abſolute or rational Quantities] the Forms are very ſimple.

1. $x=\frac{a^2}{b}$ 2. $x=\frac{ab}{c}$

The Solution of which is perform'd by finding out a third or fourth Proportional, *viz.* by ſaying thus,

1. As *b* to *a*: ſo is *a* to a third $=x$.
2. As *c* to *a*: ſo is *b* to a fourth $=x$.

Of the laſt Caſe: [Which reſpects irrational or radical Quantities] the Forms are very Simple.

1. $x^2=d^2$ Likewiſe $x^4=d^4$
Whence $x=d$ $x=d$
2. $x^2=bd$ $x_4=b^2d^2$
Whence $x=\sqrt{bd}$ $x=\sqrt{bd}$

Therefore the Solution of them conſiſts in finding the Square Root.

But in ſolving Queſtions very few Equations come out in ſuch very ſimple Forms, but for the moſt part are compounded of various Quantities, which therefore are to be further handled, and reduc'd to the moſt ſimple Form.

Here follow Examples.

In which the chief *variations* and *compound Forms* of the ſaid *Equations* (as they uſually appear) are contain'd.

And 1. In *abſolute* or *rational Quantities.*

1. $bx = 2ac$ Whence $x = \frac{2ac}{b}$

Solution: as b to $2a$: ſo is c to x
Or: as b to $2c$: ſo is a to x.

2. $3cx = b^2$ Whence $x = \frac{b^2}{3c}$

3. $2dx = 3a_2$ $x = \frac{3a^2}{2d}$

4 $cx + bx = 2b^2$ $x = \frac{2b_2}{c+b}$
As $c+b$ to $2b$: ſo b to x.

5. $3ax + bx = 6ab$ $x = \frac{6ab}{3a+b}$

6. $dx - ax = 2a^2$ $x = \frac{2a^2}{d-a}$

7. $cx + x = d^2$ $x = \frac{d^2}{c+1}$

8. $2bx=a^2+b^2$ Whence $x=\frac{a^2+b^2}{2b}$.

9. $ax=ab+b^2$ $x=\frac{ab+b^2}{a}$

10. $2cx=b^2-a^2$ $x=\frac{b-a^2}{2c}$

11. $3ax=2bc+dc$ $x=\frac{2bc+dc}{3a}$

12. $dx+cx=ab+bc$ $x=\frac{ab+bc}{d+c}$

13. $2bx-ax=a^2-\frac{1}{2}ab$ $x=\frac{a^2-\frac{1}{2}ab}{2b-a}$

14. $3ax-bx=2b^2-ac$ $x=\frac{2b^2-ac}{3a-b}$

15. $2cx=b^2+c^2-a^2$ $x=\frac{b^2+c^2-a^2}{2c}$

16. $ax-x=a^2-b^2-c^2$ $x=\frac{a^2-b^2-c^2}{a-1}$

17. $ax+cx-\frac{1}{2}dx=a^2+c^2-2d^2$ $x=\frac{a^2+c^2-2d^2}{a+c-\frac{1}{2}d}$

18. $cx=b$ $x=\frac{b}{c}$

19. $bx=4a$ $x=\frac{4a}{b}$

20. $x=ab$, or $1\,x=ab$ $x=\frac{ab}{1}$

21. $2ax=a^2-b$ $x=\frac{a^2-b}{2a}$

22. $3x=bc+d$ Whence $x=\frac{bc+d}{3}$.

23. $a^2x+b^2x=ac+bc$ $x=\frac{ac+bc}{a^2+b^2}$

24. $dcx-x=2d^2+c$ $x=\frac{2d^2+c}{dc-1}$

25. $bdx=c^2a$ $x=\frac{c^2a}{bd}$

26. $c^2x=acd+a^2b$ $x=\frac{acd+a^2b}{c^2}$

27. $2bcx-adx=3abc+bc^2$ $x=\frac{3abc+bc^2}{2bc-ad}$

28. $bfx+dhx=efg$ $x=\frac{efg}{bf+dh}$

29. $a^2x-d^2x=acd-d^2b$ $x=\frac{acd-d^2b}{a_2-d^2}$

30. $b^2dx=a^2c^2$ $x=\frac{a_2c_2}{b^2d}$

31. $2ac^2x-a^2bx=b^4+2b^2ac$ $x=\frac{b^4+2b^2ac}{2ac^2-a^2b}$

32. $4b^2x-adx=a^4-c^4-c^2a^2$ $x=\frac{a^4-c^4-c^2a^2}{4b^2-ad}$

In Irrational Quantities.

33. $2x^2=3ab$ Whence $x=\sqrt{\frac{3ab}{2}}$
or $x^2=\frac{3ab}{2}$

34. $x^2=b^2+ab$ $x=\sqrt{b^2+ab}$

35. $x^2 = b^2 - ab$
36. $4x^2 = 5a^2 - 3bc$
37. $bx^2 = 2a^2c$ $x = \sqrt{\frac{2a^2c}{b}}$
38. $ax^2 = 3b^2$
39. $bx^2 + x^2 = d^2$
40. $x^2 = a^2b$
41. $x^2 = d^3$
42. $ax^2 - 2bx^2 = ab$
43. $dx^2 = 2ac$
44. $3bx^2 = a^3 + ab^2$
45. $cx^2 + ax^2 = a^2c - b^2$
46. $b^2x^2 = 4a^3$
47. $acx^e = b^2d^2$
48. $bfx^2 + dhx^2 = e^2gh$
49. $b^2x^2 - d^2x^2 = f^2d^2$
50. $a^2x^2 - \frac{1}{2}bdx^2 = d^4 + 2abd^2$
51. $x^2 = a^2bd$
52. $2ax^2 = d^4$
53. $x^4 = 3bcdf$ $x = w\overline{3bcdf}$
54. $dx^4 + x^4 = a^2b^3$
55. $5x^4 = 8ad^3 + c^2ad$
56. $b^2x^4 = b^2c^2f^2 - 2a^6$

The Explication of ſome Caſes to the foregoing Equations.

FIrſt, as concerning the uſe of Unity, we muſt in general obſerve that in every Equation it is requir'd, that all the Quantities on both ſides be equal in number of Dimenſions; for otherwiſe they can neither bear any Proportion, nor be equal to one another. If it happen otherwiſe, that is, if an Equation ſhould occur, which has fewer Letters or Dimenſions on one ſide than on the other, or where one of the Quantities is deficient from the common Number of Dimenſions, then for the ſupplying that defect we muſt always be ſuppos'd to borrow Unity, and multiply it into the deficient Quantity as often as there ſhall be occaſion; from whence if but one Letter or Dimenſion be lacking, that Unity will repreſent a Line; if two, a Square; if three, a Cube. Wherefore by this means Equations of that kind are to be made equal every way in Number of Dimenſions, before they can be ſolv'd *geometrically*. And this (as often as it comes to paſs) is a ſign that the Problem is in it ſelf *Arithmetical*, and not to be determin'd but by Numbers: Whence it will be neceſſary, that to the given Letters or Lines certain Numbers be aſſign'd, by which at the ſame time Unity it ſelf (that is, the Line which repreſents Unity) will be determin'd. Or if Unity be aſſum'd at liberty, then the given Quantities, as to their Numerical Value, will always depend on the aſſum'd Unity, and conſequently the thing Sought will alſo depend; which therefore will be different for one or other Unity, ſo that ſometimes it will come out greater, and ſometimes leſs.

Or we may expreſs it thus; ſince, as was ſaid, Problems of that kind are in their own Nature Arithmetical, and we are to endeavour to ſolve and expound them Geometrically, it will be neceſſary that the Numerical Quantities be reduc'd to a Linear Form; by which alſo it will come to paſs that Unity itſelf will be a certain Line: Which therefore being multiply'd into another Line which repreſents ſome Number will compoſe a Square; which Square notwithſtanding will have no other than a Linear Power. *See Prop.* V. *Lin. Algeb.*

Num. 7. $cx + x = d^2$: that is, by the foregoing Obſervation $cx + 1x = d^2$: whence $x = \frac{d^2}{c + 1}$

The Diviſor $c + 1$ is an Argument that the Problem is Numerical. Whence either (1.) the Quantities c and d, if they be Lines, cannot be given or aſſum'd but under certain Numbers, by which the Unity it ſelf will at the ſame time be known: Which if it be call'd e, the Example will ſtand thus $cx + ex = d^2$, *&c.* Or (2.) if we will aſſume Unity at liberty, then that very Unity aſſum'd by the Lines c and d will determin its Numbers, *&c.*

Num. 8. $2bx = a^2 + b^2$. And $x = \frac{a^2 + b^2}{2b}$

Whence (1.) ſuppoſing $a^2 + b^2 = f^2$. Or $x = \frac{a^2}{2b} + \frac{1}{2}b = \frac{f^2}{2b}$

(by the XVII. Lin. Algeb.) it will be

as $2b$ to f: ſo is f to x.

Or (2.) As $2b$ to a: ſo is a to $\frac{a^2}{2b}$; add $\frac{1}{2}b$, you will have x.

Numb.

Num. 9. $ax=ab+b^2$. And $x=\frac{ab+b^2}{a}$

Or $x=b+\frac{b^2}{a}$

Wherefore (1.) ſuppoſing $ab+b^2=g^2$, it will be As a to g: ſo is g to x.

Or (2.) As a to b: ſo is b to $\frac{b^2}{a}$ add b, the Sum will be $=x$.

Or rather (3.) ſince the Quantity $ab+b^2$ ariſes from the multiplication of $a+b$ into b,——it will be

As a to $a+b$: ſo is b to $\frac{ab+b^2}{a}=x$.

Num. 10. $2cx=b^2-a^2$. And $x=\frac{b^2-a^2}{2c}$

1.) $b^2-a^2=g^2$. Whence, as $2c$ to g: ſo is g to x.

$$\left.\begin{array}{l} \text{2. As } 2c \text{ to } b: \text{ ſo is } b \text{ to } \frac{b^2}{2c}=d \\ \text{Likewiſe, As } 2c \text{ to } a: \text{ ſo is } a \text{ to } \frac{a^2}{2c}=f \end{array}\right\} \begin{array}{l}\text{Whence}\\ d-f=x.\end{array}$$

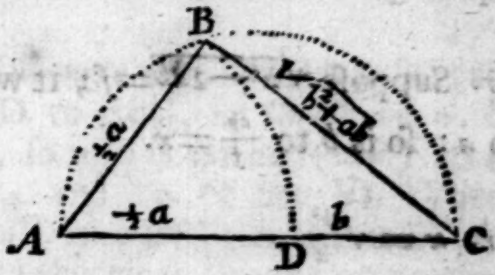
3.) By obſerving that the Quantity $b^2-{}^2$ is compounded by multiplying $b+a$ into $b-a$; it will be as $2c$ to $b+a$: ſo is $b-a$ to $\frac{b^2-a^2}{2c}=x$.

Num. 34. $x^2=b^2+ab$. And $x=\sqrt{b^2+ab}$.

1.) Supppoſe $ab=d^2$ [by the XVI. Lin. Algeb.] $x=\sqrt{b+{}^2d^2}$.

Or 2.) By obſerving the Quantity b^2+ab ariſes from the multipl. $b+a$ into b, the proportion will be as $b+a$ to x: ſo is x to b. Whence a mean Proport. between b and $b+a=x$.

Or 3.) Conſtruct a Rectangle Triangle ABC, whoſe Hypothenuſe AC may $=\frac{1}{2}a+b$ (that is, AD $=\frac{1}{2}a$, and DC$=b$) and either ſide about the right Angle [as in this place AB] $=\frac{1}{2}a$; for then the other ſide about the right Angle as BC, will $=$ $\sqrt{b^2+ab}=x$.

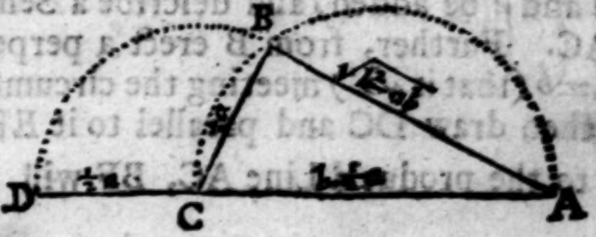


Num. 35. $x^2=b^2-ab$. And $x=\sqrt{b^2-ab}$.

1.) Suppoſe $ab=d^2$, it will be $x=\sqrt{b^2-d^2}$.

2.) As $b+a$ to x: ſo is x to $b-a$, whence a mean Proport. between $b+a$ and $b-a=x$.

3.) In the Rectangle Triangle ABC let the Hypothenuſe AC$=b-\frac{1}{2}a$, and CB$=\frac{1}{2}a$, it will be AB$=\sqrt{b^2-ab}=x$.



§. $x^2=b$. Whence $x=\sqrt{b}$.

In this place either the quantity is to be determin'd by a Number, by which Unity may be known; or if Unity be determin'd, the quantity b will depend on it, *&c.*

№. 42. $a^2x^2-2abx^2=a^2b^2$. Whence $x=\sqrt{\frac{a^2b^2}{a^2-2ab}}$

Or $ax^2-2bx^2=ab^2$. Whence $x=\sqrt{\frac{ab^2}{a-2b}}$

Or by ſuppoſing $a-2b=d$; $x=\sqrt{\frac{ab^2}{d}}$

1. *Manner*. Suppoſe $\sqrt{a^2-2ab}=f$; it will be.

As f to a: ſo is b to $\frac{ab}{f}=x$.

2. *Manner*. $x=\sqrt{\frac{ab^2}{d}}$

As d to b: ſo is b to $\frac{b^2}{d}=g$. Whence $ag=\frac{ab^2}{d}$

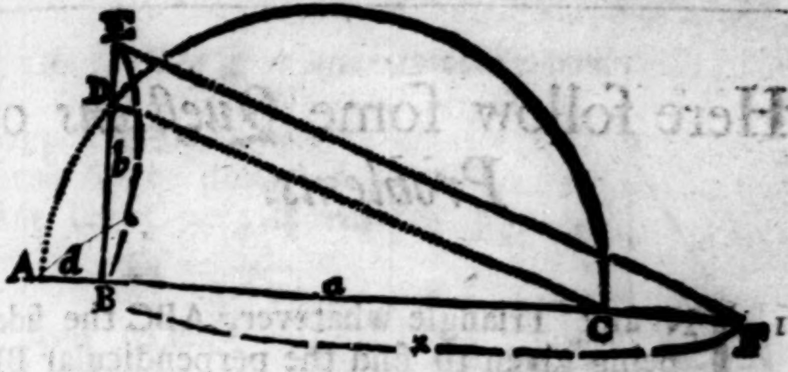
Conſequently $x^2=ag$. And $x=\sqrt{ag}$.

3. *Manner*. $dx^2=ab^2$. Whence the Proportion.

As d to a: ſo is b^2 to x^2.

The *Conſtruction:* In the Figure here drawn, let AB$=d$ [that is, $=a-2b$] and BC$=a$, (that is let d and a be added) and deſcribe a Semicircle upon AC. Further, from B erect a perpendicular BE$=b$ (that is $\sqrt{b^2}$) meeting the circumference in D, then draw DC and parallel to it EF being drawn to the produc'd Line AC, BF will $=\sqrt{\frac{ab^2}{d}}$ $=$x.

The



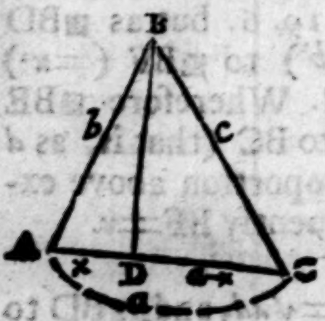
The *Demonſtration*. As AB to BC (that is d to a) ſo is ◼BD to ◼BC, by *Coroll.* 19. 6. but as ◼BD to ◼BC, ſo alſo is ◼BE, ($=b^2$) to ◼BF ($=x^2$) by the 14. and 19. of the VI. Wherefore ◼BE (that is b^2) is to ◼BF, as AB to BC (that is, as d to a) and therefore (by the proportion above explain'd) ◼BF$=x^2$, and conſequently BF$=x$.

§. Or more briefly thus: BD$=\sqrt{ad}$, and as BD to BC: ſo is BE to BF. That is, as $\sqrt{ad}$ to a: ſo is b to x, whence $x\sqrt{ad}=ab$, and $x^2ad=a^2b^2$, and $x^2d=a^2b^2$: then $x=\frac{ab}{\sqrt{ad}}$, but $\sqrt{ad}=\sqrt{a^2-2ab}$, becauſe d is ſuppos'd $=a-2b$; there returns therefore the Equation $x=\frac{ab}{\sqrt{a^2-2ab}}$, which was at firſt propos'd.

Here

Here follow ſome *Queſtions* or *Problems.*

1. IN any Triangle whatever, ABC the ſides being given to find the perpendicular BD (falling within)



Let AC $=a=14$
AB $=b=13$
BC $=c=15$
AD $=x$
∵ DC $=a-x$

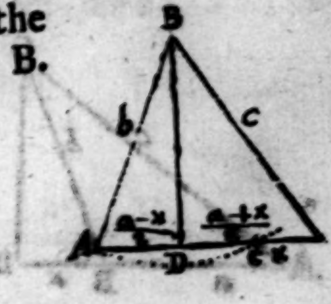
Now 1.) In the Triangle ABD
$b^2-x^2=$ the Square BD
2.) In the Triangle BCD
$c^2-a^2+2ax-x^2=$ the Square BD.

Whence $c^2-a^2+2ax-x^2=b^2-x^2$
And $2ax=b^2+a^2-c^2$
Div. $2a$

$$x=\frac{b^2+a^2-c^2}{2a}$$

The

The ſame after another manner.



Suppoſing all as before, let the Arch A*e* be deſcrib'd from B. And let *e*C$=x$; it will be

$$DC=\frac{a+x}{2}$$

And $AD=\frac{a-x}{2}$

Now 1.) In the Triangle ABD

$$b^2\frac{-a^2+2ax-x^2}{4}=BD \text{ Square}$$

2.) In the Triangle BCD.

$$\frac{c^2-a^2-2ax-x^2}{4}=BD \text{ Square}$$

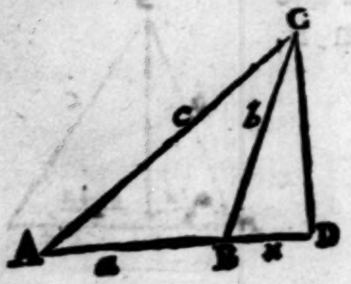
Whence $b^2\frac{-a^2+2ax-x^2}{4}=c^2\frac{-a^2-2ax-x2}{4}$

And $\frac{4ax}{4}=c^2-b^2$

That is, $ax=c^2-b^2$

$$x=\frac{c^2-b^2}{a}$$

2. In an obtuſangle Triangle ABC the ſides being given, to find the perpendicular BD (falling without.)



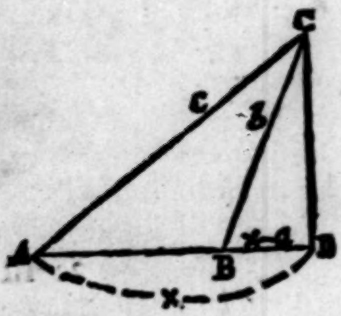
Let AB $= a = 11$
BC $= b = 13$
AC $= c = 20$
BD $= x$

Now 1.) In the Triangle BCD.
$b^2 - x^2 =$ Square CD.
2.) In the Triangle ACD.
$c^2 - a^2 - 2ax - x^2 =$ Square CD.

Wherefore $b^2 - x^2 = c^2 - a^2 - 2ax - x^2$
And $2ax = c^2 - a^2 - b^2$
Div. $2a$

$$x = \frac{c^2 - a^2 - b^2}{2a}.$$

The ſame another way.



Suppoſing the reſt as before.

Now let AD $= x$.
It will be BD $= x - a$.

Now then 1.) In the Triangle ACD

$$\left(\begin{array}{l} c^2 - x^2 = \text{Square CD.} \\ \quad 2.)\ \text{In the Triangle BCD.} \\ b^2 - x^2 + 2ax - a^2 = \text{Square CD.} \end{array}\right.$$

Whence $b^2 - x^2 + 2ax - a^2 = c^2 - x^2$

And $2ax = c^2 + a^2 - b^2$

Div. $2a$

$$x = \frac{c^2 + a^2 - b^2}{2a}$$

---

3. Two parallel right Lines AK, BL, being given, to draw a right Line DCE through the point C given within them, which may take off the Segments AD, and BE, in the ſaid Parallels in a given proportion one to another, the Points A and B being ſuppos'd to be given.

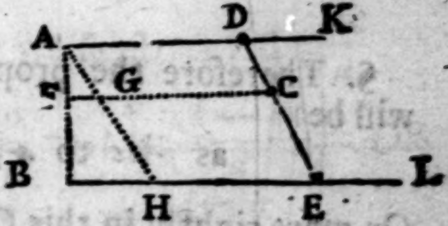
Let $AF = a$

$FC = b$

$FB = c$

$AD, GC, HE = x$

$\because FG = b - x$



Let the given *Ratio* be a to f

$\quad 10 \quad 18$

For becauſe the two Points A and B are given, we may draw the right Line AB, and CF parallel to AD or BL; as alſo AH parallel to the Line DE Sought.

Now by reaſon of the ſimilar Triangles AFG, ABH

$$\begin{array}{ccccc} AF & FG & : & AB & BH \\ a & \div\; b - x & : & a + c & \Big|\; \dfrac{ab+bc-ax-cx}{a} = BH \end{array}$$

add x=HE, it will be $\dfrac{ab+bc-ax-cx+ax}{a}$

That is $\dfrac{ab+bc-cx}{a} = BE$.

Now then for obtaining the Equation, as the given ratio: ſo is AD to BE

$$a \div f : x \div \frac{ab+bc-cx}{a}$$

there comes forth $fx = ab+bc-cx$

and $fx+cx = ab+bc$

whence $x = \dfrac{ab+bc}{f+c}$

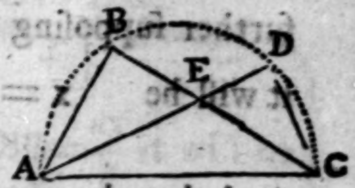
§. Therefore the proportion of the Equatio will be

as $f+c$ to $a+c$: ſo is b to x

Or more rightly in this Order:

as $c+f$ to b : ſo is $c+a$ to x:

4. Two Triangles being given upon the ſame Baſe, and in the ſame Segment of a Circle, as ABC, ADC, to find the Segment of the ſides, AD, CB.



Let $AB = a$
$BC = b$
$AD = c$
$DC = d$
$EC = x$
$\therefore BE = b - x$

Now by reaſon of the ſimilar Triangles ABE, CDE.
as AB to BE : ſo is CD to DE.

$$a \div b - x : d \mid \frac{bd - dx}{a}$$

Again for the ſame reaſon
as CD to CE : ſo is AB to AE

$$d \div x : a \mid \frac{ax}{d}$$

Now if from $AD = c$ we ſubſtract $AE = \frac{ax}{d}$

there rem. $DE = c - \frac{ax}{d}$ that is $\frac{cd - ax}{d} = DE$

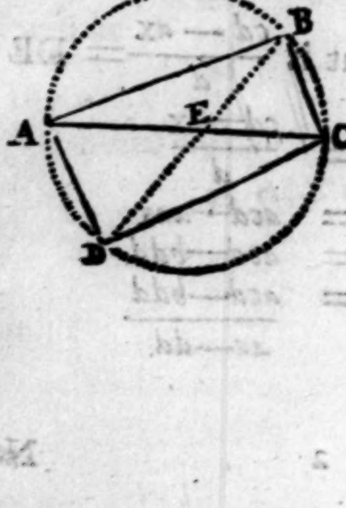
wherefore $\frac{bd - dx}{a} = \frac{cd - ax}{d}$
and further $bdd - ddx = acd - aax$
$aax - ddx = acd - bdd$
whence $x = \frac{acd - bdd}{aa - dd}$

Now (for the Conſtruction) ſuppoſing $aa=df$.
it will be $x=\frac{acd-bdd}{df-dd}=\frac{ac-bd}{f-d}$

further ſuppoſing $ac-bd=g^2$, and $f-d=h$
it will be $x=\frac{g^2}{h}$

And thus the propos'd Equation is reduc'd to the moſt ſimple form.

5. Two Triangles being given upon the ſame Baſe, and in oppoſite Segments of a Circle, as ACB, ACD, and the Diagonal BD being drawn and given, to find the Segments, AE, EC; BE, ED.



Let $AB=a$
$AC=b$
$BD=c$
$CD=d$
$EC=x$
$\therefore AE=b-x$

Now by reaſon of the ſim. Triangles ABE, DEC.

1.) As DC to CE : ſo is AB to BE

$$d \;∹\; x \;:\; a \;\Big|\; \frac{ax}{d}$$

2.) As AB to DE : ſo is DC to DE

$$a \;∹\; b\text{-}x \;:\; d \;\Big|\; \frac{db-dx}{a}$$

Now if from BD$=c$ we ſubtr. BE$=\frac{ax}{d}$ it will be

$$DE = c - \frac{ax}{d} = \frac{cd-ax}{d}$$

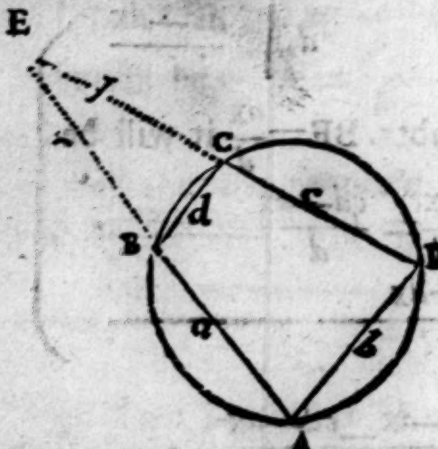
But alſo $DE = \frac{db-dx}{a}$

Wherefore $\frac{db-dx}{a} = \frac{cd-ax}{d}$

And laſtly $x = \frac{acd-bdd}{aa-dd}$

Altogether as in the foregoing.

6. Four right Lines being given, to find the Circle in which they are the ſides of a quadrilateral Figure.



Let AB $= a$
AD $= b$
CD $= c$
BC $= d$
BE $= x$
CE $= y$

For the difficulty of the Problem lies in finding BE and CE.

Now then by reaſon of the ſimilar Triangles AED, CEB, it will by 22. III.

1.) As AD to AE : ſo is CB to CE

$$b \;\underset{\cdot\cdot}{-}\; a+x \;:\; d \;\underset{\cdot\cdot}{-}\; y$$

Whence $by = ad+dx$

and $y = \frac{ad+dx}{b}$

2.) As AD to DE : ſo is CB to BE

$$b \;\underset{\cdot\cdot}{-}\; c+y \;:\; d \;\underset{\cdot\cdot}{-}\; x$$

Whence $cd+dy = bx$

and $y = \frac{bx-cd}{d}$

Wherefore $\frac{bx-cd}{d}=\frac{ad+dx}{b}$

And further $b^2x-bcd=ad^2+d^2x$

$b^2x-d^2x=ad^2+bcd$

$x\ =\frac{ad^2+bcd}{b^2-d^2}$

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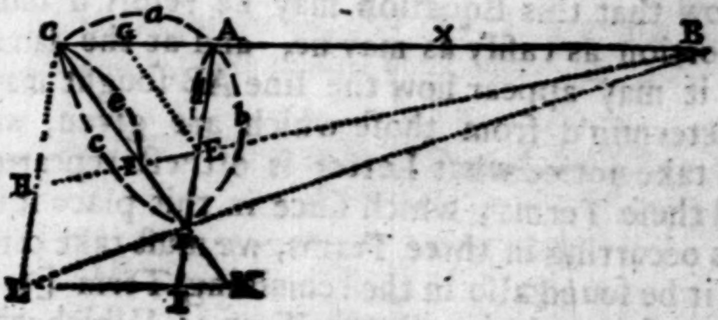
7. To find the diſtance of two Towers, A, B, there being given $AC=a$, $AD=b$, $CD=c$, $AE=d$ $CF=e$.



Let $AB=x$

Now GE being drawn parallel to CD, by reaſon of the ſimilar Triangles ADC, AEG, 'twill be

1.) As AD to DC : ſo is AE to EG.

$b \mathrel{::} c : d \mid \frac{cd}{b}$

2.) As AD to AC : ſo is AE to AG

$b \mathrel{::} a : d \mid \frac{ad}{b}$

add $AB=x$ it will be $GB=\frac{ad+bx}{b}$

Now 3.) By reaſon of the ſimilar Triangles CBF, GBE

As CF to CB : ſo is GE to GB

$$e \;\overline{\cdot\cdot}\; a+x : \frac{cd}{b} \;\overline{\cdot\cdot}\; \frac{ad+bx}{b}$$

Whence $ade+bex = acd+cdx$
and $bex-cdx = acd-ade$
and $x = \frac{acd-ade}{be-cd}$

Now that this Equation may be reſolv'd into Proportion as eaſily as may be, and at the ſame time it may appear how the line AB ſought may be determin'd from thoſe which are given, we muſt take notice what Letter is oftneſt repeated in all theſe Terms; which ſince in this place it is d, as occurring in three Terms, we muſt take care that it be found alſo in the remaining Term [be] and conſequently in all the Terms: Which will be done, if the Plane be be chang'd into another, one of whoſe ſides $=d$, that is by ſaying,

As d to b : ſo is e to the other ſide of this Plane, which ſide may be call'd f, whence $df=be$.

and conſequently $x = \frac{acd-ade}{df-cd}$

that is, ſtriking out d: $x = \frac{ac-ae}{f-c}$

further ſuppoſ. $c-e=g$. and $f-c=h$.
it will be $x = \frac{ag}{h}$

and therefore as h to a : ſo is g to x.

The

The ſame another way.

For the ſame Fraction $\frac{acd-ade}{be-cd}$ may be yet reſolv'd after another manner: that is, by conſidering, that in two terms, *cd* is found, and in the other two, *e* is found: therefore if the Plane *cd* be chang'd into another, one of whoſe ſides is *e*, the Letter *e* will be found in all the terms, and conſequently ſtruck out.

Let it therefore be, as *e* to *c*, ſo is *d* to a fourth which call *k*, and it will be $ek=cd$.

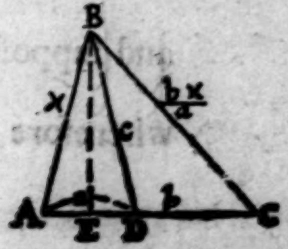
and therefore $\frac{aek-ade}{be-ek}=\frac{acd-ade}{be-cd}$

that is $\frac{ak-ad}{b-k}=x$, &c.

---

8. In the Triangle ABC, the Angle B being biſected by the right Line BD, there are given AD, DC, and BD; Requir'd the ſides AB, BC.

Let $AD=a$
$DC=b$
$BD=c$
$AB=x$



Now 1.) by 3 of the VI. it will be,

As AD to DC : ſo is AB to BC

$$a \;::\; b \;:\; x \;\Big|\; \frac{bx}{a}$$

2.) by 13. II.

$$\square AD + \square BD - \square AB.$$

$$\frac{a^2 + c^2 - x^2}{2a} = DE$$

3.) by 12. II.

$$\square BC - \square BD - \square DE.$$

$$\frac{\frac{b^2x^2}{a^2} - c^2 - b^2}{2b} = DE$$

that is $\dfrac{b^2x^2 - a^2c^2 - a^2b^2}{2ba^2} = DE$

whence $\dfrac{b^2x^2 - a^2c^2 - a^2b^2}{2ba^2} = \dfrac{a^2 + c^2 - x^2}{2a}$

that is $\dfrac{b^2x^2 - a^2c^2 - a^2b^2}{ab} = a^2 + c^2 - x^2$

and $b^2x^2 - a^2c^2 - a^2b^2 = a^3b + abc^2 - abx^2$

then $b^2x^2 + abx^2 = a^2b^2 + a^2c^2 + abc^2 + a^3b$

And dividing both ſides by $b+a$

$$bx^2 = a^2b + ac^2$$

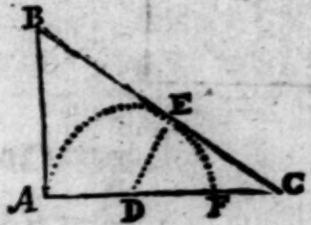
whence $x^2 = \dfrac{a^2b + ac^2}{b}$

and ſuppoſe $c^2 = bd$

$$x^2 = a^2 + ad$$

wherefore $x^2 = \sqrt{a^2 + ad}$

9. In the right angled Triangle ABC, having deſcribed the Semicircle AEF, which touches the ſides AB and BC, we are to find the ſide AB, from the given Lines AC and DE, which is perpendicular to BC from the center D.



Let $AC = a$
$DE = b$
$AB = x$
$\because FC = a-2b$

Now 1.) By reaſon of the ſimilar Triangles BAC, DEC.

As AB to AC : ſo is DE to EC.

$$x \div a : b \mid \frac{ab}{x}$$

2.) By 36. III. $\blacksquare EC = \square ACF$.

that is, $\dfrac{a^2b^2}{x^2} = a^2-2ab$

and $a^2b^2 = a^2x^2-2abx^2$.

that is, $ab^2 = ax^2-2bx^2$.

$\dfrac{ab^2}{a-2b} = x^2$, &c.

The reſolution of which Equation may be ſeen among the *formulæ* mention'd before, *N*°. 42

10. Requir'd a Line whoſe Square, together with a Rectangle under the whole, or a third part of it, may be equal to a given Plane $= a^2$.

Let the Line requir'd $= x$

$3 = b$

$\therefore \frac{1}{3}$ of the Line $= \frac{x}{b}$

Wherefore $x^2 + \frac{x^2}{b} = a^2$

$$bx^2 + x^2 = ba^2$$

Divide $b + 1$

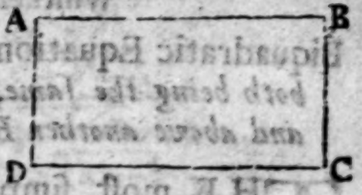
$$x^2 = \frac{ba^2}{b+1}$$

And ſuppoſing $1 = e$ it will be

$$bx^2 + ex^2 = ba^2$$

$$x^2 = \frac{ba^2}{b+e}$$

11. Let there be a Rectangle AC, whoſe length AB is double its breadth AD, and the Sum of the Squares of AB and AD is ten times the Sum of their Sides: Requir'd the ſaid Sides AB, AD.



That the Problem may be more generally ſolv'd, let AD be to AB, as *e* is to *f*; and ■ AD + ■ AB to AD + AB, as *g* is to *e*.

Now let $AD = x$

it will be $AB = \frac{fx}{e}$

and the Sum of both $= \frac{ex + fx}{e}$

but the Sum of the Squares $= x^2 + \frac{f^2x^2}{e^2}$

or $\frac{e^2x^2 + f^2x^2}{e^2}$

whence from the other ſuppoſition it will be as

g to e: ſo is $\frac{e^2x^2 + f^2x^2}{e^2}$ to $\frac{ex + fx}{e}$

Whence $\frac{e^3x^2 + ef^2x^2}{e^2} = egx + fgx$

that is $e^2x^2 + f^2x^2 = egx + fgx$

div. $\underline{x}$

$$x = \frac{eg + fg}{e^2 + f^2}$$

CLAS-

Classis II.

Of Quadratic Æquations, or *thoſe of two Dimenſions* :

Whither alſo are referr'd

Biquadratic Equations *alike affected ; the manner of both being the ſame, except that here there is over and above another Extraction of the Root.*

THE moſt ſimple Caſes or Forms of this Claſs are three :

$$1.)\quad x^2 = ax + b^2$$
$$2.)\quad x^2 = ax + b^2$$
$$3.)\quad x^2 = ax - b^2$$

So in Biquadratics.

$$1.)\quad x^4 = a^2x^2 + b^4$$
$$2.)\quad x^4 = a^2x^2 + b^4$$
$$3.)\quad x^4 = a^2x^2 - b^4$$

The Solutions of which ſeveral *Formulæ* are thus

$$1.)\quad x = \sqrt{\tfrac{1}{4}a^2 + b^2} + \tfrac{1}{2}a$$
$$2.)\quad x = \sqrt{\tfrac{1}{4}a^2 + b^2} - \tfrac{1}{2}a$$
$$3.)\quad x = \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}a^2 - b^2}$$

Where Note, That this third *Formula* has two true Roots, the greater of which $x = = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}$, and the leſs $x = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2}$

But Uſe and Practiſe will tell us which of them is to be apply'd in any Queſtion propos'd; but oftentimes both of them will be uſeful in the Conſtruction.

So in Biquadratics.

1.) $x^2 = \sqrt{\frac{1}{4}a^4 + b^4} + \frac{1}{2}a^2.$

$x = \sqrt{\sqrt{\frac{1}{4}a^4 + b^4} + \frac{1}{2}a^2}$

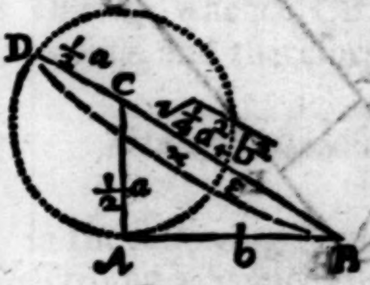
2.) $x^2 = \sqrt{\frac{1}{4}a^4 + b^4} - \frac{1}{2}a^2$

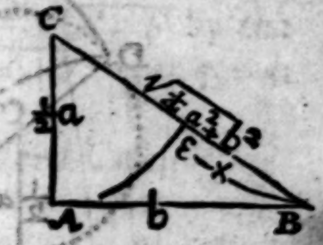
$x = \sqrt{\sqrt{\frac{1}{4}a^4 + b^4} - \frac{1}{2}a^2}$

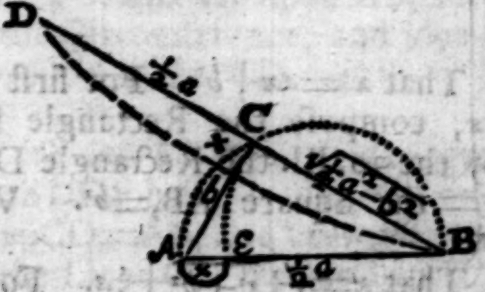
3.) $x^2 = \frac{1}{2}a^2 \pm \sqrt{\frac{1}{4}a^4 - b^4}$

$x = \sqrt{\frac{1}{2}a^2 \pm \sqrt{\frac{1}{4}a^4 - b^4}}$

Farther, *the Conſtructions of the ſeveral Caſes* among the reſt are theſe.



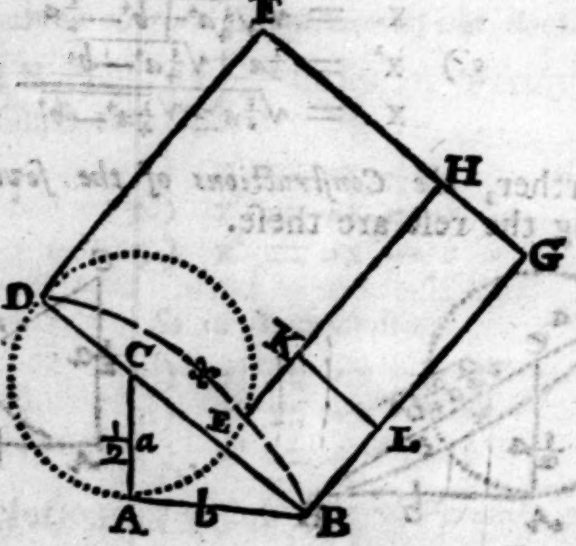




The Explication and Demonſtration of which, we ſhall ſee by what follows.

Demonſtration of the firſt Caſe.

Let AC$=\frac{1}{2}a$, and AB$=b$; which being ſet at right Angles in A, upon the Center C, with the Radius CA, deſcribe the Circle ADE, and drawing a right Line DCB$=x$. make upon it the Square BDFG$=x^2$; draw farther EH parallel to BG.



I ſay 1.) That $x^2=ax+b^2$. For firſt DCE$=a$, and DF$=x$, compoſe the Rectangle DH$=ax$; and then by the 36. III. the Rectangle DB*e* [that is, GBE] $=$ the Square AB, $=b^2$. Wherefore $ax+b^2=x^2$.

I ſay 2.) That $x=\sqrt{\frac{1}{4}a^2+b^2}+\frac{1}{2}a$. For the Squ. AC$=\frac{1}{4}a^2$; and ▨ AB$=b^2$; to which (by the 47. I.) ▨ BC is equal; therefore BC$=\sqrt{\frac{1}{4}a^2+b^2}$. Add CE$=CA=\frac{1}{2}a$ it will be BD$=$BC$+$CD. That is $x=\sqrt{\frac{1}{4}a+b^2}+\frac{1}{2}a$. *q. e. d.*

De-

Demonſtration of the Second Caſe.

The Figure being conſtructed as before, now let *e*B=x; whence DB=$a+x$, and the Square *e*L deſcrib'd upon *e*B, =x^2.

I ſay now 1.) $x^2=b^2-ax$. For GB=DB=$a+x$ & LB=*e*B=x; and therefore GL=D*e*=a; which with KL=*e*B=x, conſtitutes ▭ GK=ax. Therefore ▭ G*e* [that is ▭ GK+▨*e*L] =x^2+ax. Which ſame ▭ G*e*, ſince it is ſhewn in the foregoing Caſe = ▨AB=b^2; it will be $x^2+ax=b^2$ and $x^2=b^2-ax$.

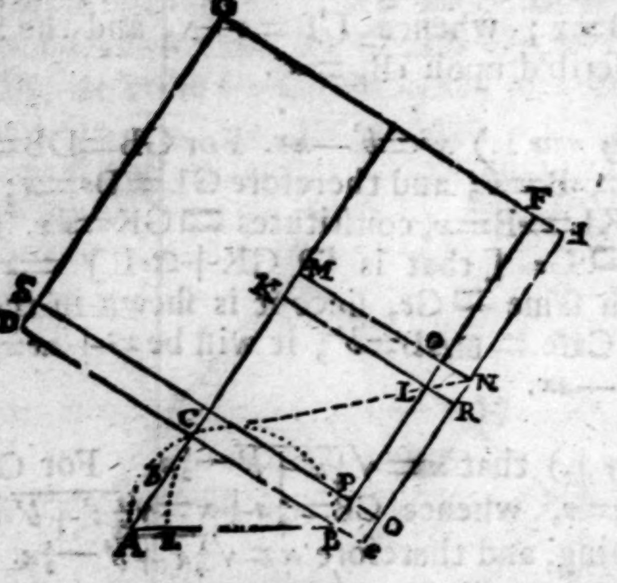
I ſay 2.) that $x=\sqrt{\frac{1}{4}a^2+b^2}-\frac{1}{2}a$. For C*e*=$\frac{1}{2}a$, & *e*B=$x$, whence CB=$\frac{1}{2}a+x=\sqrt{\frac{1}{4}a^2+b^2}$ by the foregoing, and therefore $x=\sqrt{\frac{1}{4}a^2+b^2}-\frac{1}{2}a$. *q. e. d.*

Demonſtration of the Third Caſe.

In the following Figure let AB=$\frac{1}{2}a$, & AC=b. Deſcribing a Semicircle upon AB, let AC be apply'd to the Circumference, and joyn BC: which being produc'd make CD=AB.

Now (1.) let DB=x, as the greater Root, upon which ◻DF being made, will =x^2. Further make CE=CD=AB, and draw *e*H parallel to BF, meeting the Lines GF produc'd in H, it will be ▭ DH=ax.

I ſay 1.) that $x=\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-b^2}$ by 47. 1 and the conſtruction.



I ſay 2.) that $x^2=ax-b^2$. For the Square CN being made upon the ſide $Ce=\frac{1}{2}a$, it will $=\frac{1}{4}a^2$: and likewiſe upon $CB=\sqrt{\frac{1}{4}a^2-b^2}$, deſcribing the Square CL, it will $=\frac{1}{4}a^2-b^2$. Wherefore the Gnomon KNB will $=b^2$: but ▭BH is made equal to this Gnomon, which is ſhewn thus: The Diagonal CN being drawn, the Complements ML, eL, will be equal by 43. I. and to eL is = ▭OH. For if from BF=BD be taken away BO$=e$N$=e$C, there will remain OF=BC,=BL: Whence ſince alſo OH=OK, ▭BH will = the Gnomon KNB $=b^2$; and conſequently □DF=▭DH−▭BH; that is, $x^2=ax-b^2$. *q. e. d.*

Let (2r) Br, = AE, = x, as being the leſs Root: upon which the Square BQ being made will $= x^2$, and ▭ eS will $= ax$; from which if we take away ■ BQ, the remaining ▭ BS will = BH $= b^2$. Whence the reſt is eaſily underſtood.

Further, *the Compound Forms of Quadratic and Bi-quadratic Equations* are commonly of this kind, to be reduc'd to ſimple ones after the ſame manner as in the firſt Claſs.

1. $2x^2 = 4ax - a^2$. And $x^2 = 2ax - \frac{1}{2}a^2$.

2. $x^2 = 2ax + 2bx - ab$.
   by ſuppoſing $2a + 2b = f$, it will be
   $x^2 = fx - ab$.

3. $2x^2 = 2ax + b^2 - a^2$.
   ſuppoſing $b^2 - a^2 = d^2$ it will be
   $x^2 = ax + \frac{1}{2}d^2$

4. $ex^2 = 2adx + 2adf$
   $x^2 = \frac{2adx + 2adf}{e}$ and ſuppoſ. $\frac{2ad}{e} = g$
   $x^2 = gx + gf$.

5. $bx^2 = abx + ac^2$
   $x^2 = ax + \frac{ac^2}{b}$
   ſuppoſ. $\frac{c^2}{b} = d$, 'twill be $\frac{ac^2}{b} = ad$, whence
   $x^2 = ax + ad$

6. $3ax^2 = b^2x + ab^2$

$x^2 = \frac{b^2x}{3a} + \frac{b^2}{3} \mid \frac{b^2}{3a} = d$; whence

$x^2 = dx + \frac{1}{3}b^2$

7. $bx^2 = -d^2x + abf - ad^2$

$x^2 = -\frac{d^2x + abf - ad^2}{b} \mid \frac{d^2}{b} = g$; whence

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$x^2 = -gx + af - ag \mid f - g = k$

$x^2 = -gx + ak$, &c.

8. $5x^2 = 9ax + 9bx + 2b^2 - 4\frac{1}{2}a^2$

$\frac{9a + 9b}{5} = d \mid \frac{2b^2 - 4\frac{1}{2}a^2}{5} = \pm f^2$

$x^2 = dx \pm f^2$

9. $ax^2 - bx^2 = -a^2x + b^3$

$a - b = d \mid \frac{a^2}{d} = f \mid \frac{b^3}{d} = bg$; whence

$x^2 = -fx + bg$

10. $4cx^2 = a^2x + b^2x - 2abx + 4c^2x + 4ab^2$, in this place we muſt take notice that $a^2 + b^2 - 2ab$ is a rational Square, whoſe Root $a - b$ being ſuppos'd $= d$, then will be

$4cx^2 = d^2x + 4c^2x + 4ab^2$, and

$x^2 = \frac{d^2x}{4c} + cx + \frac{ab^2}{c}$ further

ſuppoſ. $\frac{d^2}{4c} = f \mid f + c = g \mid \frac{ab^2}{c} = ak$ 'twill be

$x^2 = gx + ak$.

11. $x^2=\frac{rkx+rm^2}{e+f+g+b=n}\,\Big|\,\frac{rk}{n}=p\,\Big|\,\frac{rm}{n}=q$; whence

$x^2=px-mq$.

12. $b^2x^2=-a^2cx+a^2b^2$

$x^2=-\frac{a^2cx}{b^2}+a^2\,\Big|\,\frac{a^2}{b}=d\,\Big|\,\frac{dc}{b}=f$; whence

$x^2=-fx+a^2$

13. $b^2x^2=2ac^2x-a^2c^2$

$-d^2b\quad -\frac{1}{4}d^4$, &c.

14. $x^2=-\frac{2bc^2x+b^2c^2+a^2d^2-a^2c^2}{f^2}$

$\frac{2bc^2}{f^2}=g\,\Big|\,\frac{b^2c^2}{f^2}=b^2\,\Big|\,\frac{a^2d^2}{f^2}=k^2$

$\frac{a^2c^2}{f^2}=l^2\,\Big|\,b^2+k^2-l^2=m^2$; whence

$x^2=-gx+m^2$

15. $b^2x^2-a^2x^2=2a^2bx-2b^3x+a^2b^2$

$a^2=bd$; whence

$bx^2-dx^2=2a^2x-2b^2x+a^2b$

$b-d=f\,\Big|\,\frac{2a^2-2b^2}{f}=g\,\Big|\,\frac{a^2}{f}=b$; whence

$x^2=gx+bb$

16. $a^2x^2+b^2x^2=f^2bx+a^2c^2-\frac{1}{4}f^4$

$a^2+b^2=g^2$; whence

$x^2=\frac{f^2bx+a^2c^2-\frac{1}{4}f^4}{g^2}$.

$\frac{f^2}{g} = b \Big| \frac{a^2}{g} = k$; whence

$$x^2 = \frac{bbx + kc^2 - \frac{1}{4}bf^2}{g}$$

again $\frac{bb}{g} = l \Big| \frac{c^2}{g} = m \Big| \frac{f^2}{g} = b$; whence

$x^2 = lx + km - \frac{1}{2}b^2$

$km - \frac{1}{2}b^2 = n^2$; whence

$x^2 = lx \pm n^2$

17. $d^2x^2 - c^2x^2 = -c^2bx + d^2a^2 - c^2a^2 + b^2c^2$

first divide $d^2a^2 - c^2a^2$ by $d^2 - c^2$
The Quotient will be a^2, and ſuppoſing $d^2 - c^2 = f^2$, it will be

$$x^2 = -\frac{c^2bx + b^2c^2}{f^2} + a^2$$

further $\frac{c^2}{f} = g$, it will be

$x^2 = -\frac{gbx + b^2g}{f}$; at length $\frac{bg}{f} = b$; it will be

$x^2 = -bx + bb$

18. $b^2x^2 - 4x^2 = -8ax + 4a^2 + b^4$, &c.

Here we might ſubjoin the Conſtructions of Quadratick Forms, but what has been ſpoken of before, *Prop.* 25. *Lin. Alg.* and what ſhall be ſaid in the following Problems, ſeem ſufficient for this matter.

Queſtions and *Examples*

For adfected

Quadratick EQUATIONS.

Probl. 1. *TO divide the given Quantity into* x *the greater and* y *the leſs part, ſo that the Product under the Parts may be equal to* b *given.*

The Analyſis. Becauſe firſt by hyp. $x+y=a$ it will be $y=a-x$. And ſecondly, by hyp. $xy=b$, therefore it will be $y=\frac{b}{x}$, therefore ſince $a-x=y$ $(y=)\ \frac{b}{x}$ it will be $ax-x^2=b$, or $x^2=ax-b$; Wherefore from the Premiſſes it will be $x=\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}$. And $y=\frac{a}{2}-\sqrt{\frac{a}{4}-b}$.

Another way. Becauſe $a-x=y$, being multiply'd on both ſides by x, it will be $ax-x^2=xy=b$. (by hyp.) Wherefore $ax-x_2=b$, &c.

Another way yet. The leſs part by hyp. will $a-x$, which if we multiply into the greater x, the Product will be $ax-x^2=b$ given, by hypothesis

Note. By the ſame methods are Numbers as well as Magnitudes diſcover'd.

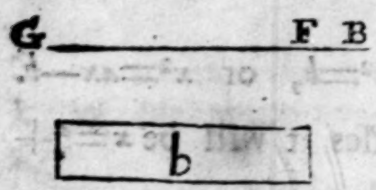
An Example in Numbers.

Suppoſe that $a=20$, and the Product $b=36$, and $x=$ the greater Number Sought, it will be $20-x=$ the leſs, and conſequently x into $20-x$, or $20x-x^2=36$, and conſequently $x^2=20x-36$. Whence the Numbers Sought are found out from the Premiſſes, *viz.*

$$x=10+(\sqrt{100-36}=)\ 8=18$$

and $$y=10-(\sqrt{100-36}=)\ 8=2$$

Example in Lines.



Suppoſe BG be divided in F, ſo that the Rectangle BF+FG may be equal to the given Rectangle b.

To find this. Suppoſe $BG=a$, the greater Segment $FG=x$, therefore the leſs Segment BF will $=a-x$, whence the Rectangle under the Segments x in $a-x$, or $xa-x^2=b$.

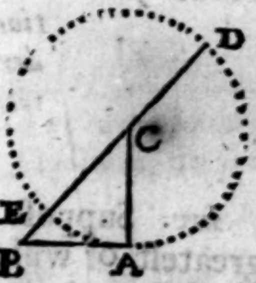
Whence as before $x=\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}$.

and $y=\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}$.

The Demonſtration of all which is done at firſt ſight. For adding and multiplying thoſe known quantities which are found equal to x and y the unknown, you will have both the Sum and the Product a and b, which was the thing propos'd.

This eaſie Problem is drawn out more at large for the ſake of young Beginners, both that the way to what comes after may be leſs intangled, and alſo may ſerve for explaining the reſt. For which reaſon, we ſhall not think it amiſs to joyn the Geometrical Conſtruction and Demonſtration of it.

Conſtruction.



Make AB a mean proportional between the Sides of the Rectangle b; on which as a Baſe and with Hypothenuſe $BC = \frac{GB}{2}$ make a right angl'd Triangle ABC; laſtly, on the Center C, with the Radius CA, deſcribe the Circle AED. I ſay that BD and BE are the Segments ſought.

Demonſt. 1. By the Conſtruction $2BC = GB$. But it appears that $2BC = BD + BE$; therefore BE and BD the two Sides found, are together equal to BG given.

2. By the Conſtruction $BA^2 = b$, but (by III. 36 of *Eucl.*) $BA^2 = BE + BD$, therefore the Rectangle under the Sides found is equal to b given. *Q. E. D.*

Probl. 2. *The Product and Difference of two Quantities being given, to find them ſeverally.*

Probl. 3. *The Difference of two Quantities, and the Sum of their Squares being given, to find the Quantities.*

Probl. 4. *To find two Quantities, from their Product, and the Sum of their Cubes given.*

Probl. 5. *From the Sum of three continued Proportionals, and the Sum of the Squares under their Extremes, to find the Proportionals.*

Inv. Suppoſe x, y, and z, be propos'd, the greateſt of which is x, the Sum of the Proportionals a, and the Sum of the Squares b. Becauſe therefore x, y, z ∺ it will be $y^2 = zx$ but by hypotheſis $x^2 + z^2 = b$, wherefore adding on one ſide $2y^2$, on the other $2zx$ it will be $x^2 + 2zx + z^2 = b + 2y^2$, and conſequently $x + z = \sqrt{b + 2y^2}$. Moreover by hypoth. $a = y + (z + x =) \sqrt{b + 2y^2}$, whence $a - y = \sqrt{b + 2y^2}$, and $a^2 - 2ay + y^2 = b + 2y^2$ & $y^2 = -2ay - b + a^2$. Wherefore by the Premiſſes it will $y = \sqrt{2a^2 - b} - a$, &c.

Probl. 6. *The one Extreme of three continued Proportionals, and the Sum of the Squares under the remaining, being given, to find the Proportionals.*

Suppoſe a, x, y ∻ and the Sum of the Squares b. It will be by hypoth. $ay=x^2$ therefore adding on both Sides y^2: it will be $y^2+ay=x^2+y^2=b$, therefore $y=\sqrt{\frac{a^2}{4}+b}-\frac{a}{2}$

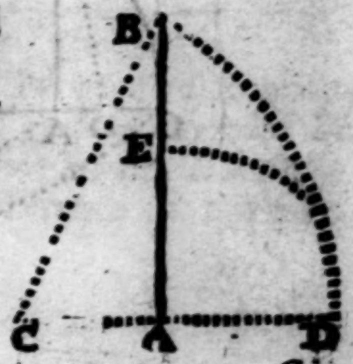
Probl. 7. *The Sums or Differences of the Extremes, and means of four continued Proportionals being given, to find out the Terms ſeverally.*

Probl. 8. *Let there be given the Sum of four continued Proportionals, and of their Squares under the Means or Extremes, to find them ſeverally.*

Probl. 9. *To cut a given Line* AB, *ſo that the Square of the part* AE *may be equal to the Rectangle under the whole and the other part.*

Inv. Suppoſe $AB=a$, and $AE=x$, the other part EB will $=a-x$; and conſequently by hyp. $AE^2=AB$ in EB or $x^2=a$ in $a-x=a-ax$, and therefore

$x=\sqrt[2]{+\frac{a^2}{4}}-\frac{a}{2}$

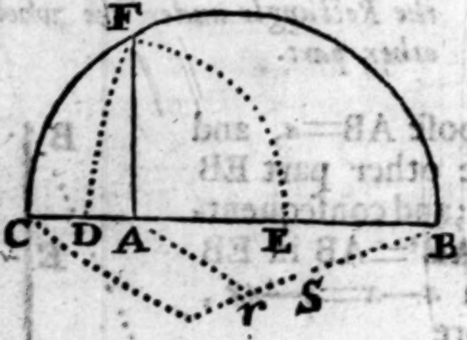


Conſtruction. Let CA the half of AB be ſet perpendicular to it, and $CD=CB$, and $AE=AD$. I ſay it is done.

Demonſtration. $CA^2+AB^2=CB^2=CD^2=CA^2+$ (AD) $EA\times$ (2CA) $AB+$ (AD^2) AE^2, therefore taking away on both ſides CA^2+AE in AB it will be $AE^2=AB^2=AE$ in $AB=AB\times(AB-AE)$ BE. therefore $AE^2=BB$ in EB. *Q. E. D.*

Probl. 10. *To cut a given Line* AB, *ſo that the Square of the part* AE *may have a given Proportion, as* r *to* s, *to the Rectangle under the whole and the other part.*

Invention. Let there be given $AB=a$ the Part ſought $AE=x$ will be the other part $EB=a-x$.
and by hyp. AE^2. $AB\times E :: r.s.$
i. e. x^2 (a in $a-x$) $a^2-ax :: r.s.$



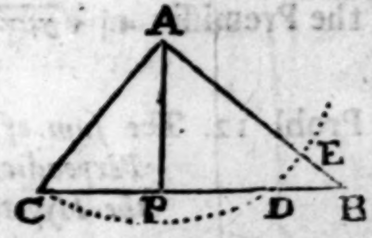
Conſtruction 1. Let there be, as in the Sheme *s.* *r.* :: AB. AC=*t*. Therefore AC$=\frac{r}{s}$ in AB. And $x^2=ta-tx$.

Whence $x=\sqrt{ta+\frac{tt}{4}}-\frac{t}{2}$

Conſtruction 2. Having deſcrib'd a Semicircle upon BA+AC, apply AF at right Angles to the Diameter, and let DA=DC and DE=DF. I ſay, that AE is the Segment Sought.

Demonſtration. DA²+ (AF²) CA in AB=DF² =DE²=DA²+AE²+(2DA) CA in AE. Therefore (CA in AB—CA in AE) AC in (AB—AE) EB=AE²; but by Conſtr. 1. AC$=\frac{r}{s}$ in AB,

Therefore $\frac{r}{s}$ + AB in EB=AE², and conſequently *r. s* :: AE² AB in EB. *Q. E. D.*



Note. That in the following Problem there is ſuppos'd a Rectangle Triangle, whoſe Hypotheneuſe is noted by the Letters BC, the greater ſide by BA; the leſs by CA, the Perpendicular upon the Hypotenuſe by AP, the greater Segment of the Hypotenuſe by PB, the leſs by CP, and their difference by DB, but the difference of the ſides by EB.

Lemma 1. That the Square of the Hypotenuſe be equal to the Squares under the ſides taken together. *Eucl.* I. 48. or $BC^2 = BA^2 + CA^2$.

Lemma 2. That ſimilar Triangles, ſuch as BAC, BPA, APC, have their ſides proportional which lie over againſt equal Angles. *Eucl.* VI. *p.* 8. def. 1.

Probl. 11. *The difference of the ſides* EB, *and the Perpendicular upon the Hypotenuſe* AP *being given, to find the Hypotenuſe* BC, *&c.*

The *Analyſis.* Let $AP = a$, $EB = b$, $CB = x$, and $AC = y$. By *Lemma* 1. it will be $x^2 = y^2 + (\overline{y+b}^2 =)$ $y^2 + 2yb + b^2 = 2y^2 + 2yb + b^2$. Therefore $x^2 - b^2 = 2y^2 + 2yb$.

And by *Lemma* 2. BC . CA :: BA . AP.
or $x \;.\; y :: y+b \;.\; a$
and therefore $ax = y^2 + yb$, therefore $2ax = (2y^2 + 2yb =)\ x^2 - b^2$. Whence $x^2 = 2ax + b^2$, and from the Premiſſes $a + \sqrt{b^2 + a^2} = x$. *Q. E. D.*

Probl. 12. *The ſum of the ſides* AB+AC, *and the Perpendicular* AP *being given to find the Hypothenuſe* BC, *&c.*

Probl. 13. *To find the Hyp. and the Triangle from the given difference of the Segments* DB, *and either of the ſides.*

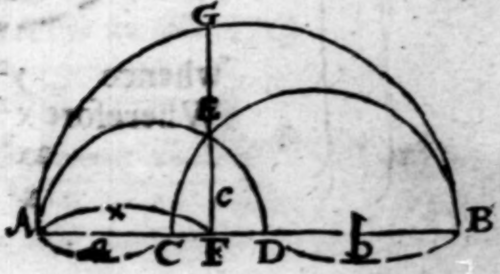
Probl. 14. *To conſtruct a Triangle, from the given difference of the Segments* DB, *and the Sum of the ſides* AB+AC.

Probl. 15. *From the given difference of the Segments* DB, *and the difference of the ſides* EB, *to find the reſt.*

Probl. 16. *From either of the ſides, and the alternate Segment to find the Triangle.* Note, *If* AB *be given,* PB *will be called the alternate Segment.*

Probl. 17. *The difference between the Hypotenuſe and the ſides being given,* viz. CB—CA *and* CB—BA, *to find the Hypotenuſe,* &c.

Prob. 18. *In the Figure plac'd under are given* AC = *a*, BD = b, FE = c, *Requir'd the Diameters of the ſeveral Circles.*



Let AF = x

∵ CF = x — *a*

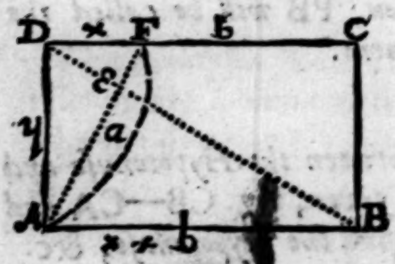
The finding out of the Equation is eaſy.

And there will come out $x^2 = ax + \frac{ac^2}{b}$

and ſuppoſing $\frac{c^2}{b} = d$, it will be

$$x^2 = ax + ad.$$

Probl. 19. *In the Rectangle* ABCD, *drawing the Line* AF *at right Angles through the Diagonal* BD *are given* AF = a, and FC = b; *Requir'd the ſides of the Rectangle.*



Let DF = x
DA = y

Now 1.) In the Triangle ADF

$$y^2 = a^2 - x^2$$

2.) Becauſe △ADF, BAD are ſimilar.

$$y \mathrel{\underset{\cdot\cdot}{-}} x : x + b \mathrel{\underset{\cdot\cdot}{-}} y$$

whence $y^2 = x^2 + bx$

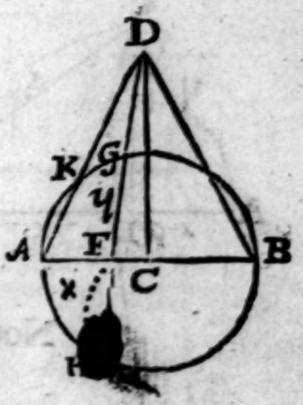
Wherefore $x^2 + bx = a^2 - x^2$

$$2x^2 = -bx + a^2$$

$$x^2 = \frac{-bx + a^2}{2}$$

$$x = \sqrt{\tfrac{1}{16}b^2 + \tfrac{1}{2}a^2} + \tfrac{1}{4}b$$

Probl. 20. *Upon the Baſe of an equilateral Triangle* ABD *is deſcrib'd a Circle, and a right Line* DH *from the top* D, *drawn thro' a given point of the Diameter* F, *cutting the circumference in* G; *requir'd the length of the ſecant Line* DH, *and its ſeveral Segments, as* DG, GF, FH; *ſuppoſing that the Diameter of the Circle is given.*



Let AC, AK, KD $= a$
CF $= b$
$\because$ AF $= a - b$
$\because$ BF $= a + b$
DC $[= \sqrt{3a^2}] = c$
$\because$ DF $= \sqrt{b^2 + c^2}, = d$
FH $= x$
FG $= y$
$\because$ DG $= d - y$

Now 1.) By 36. III.
▭ HDG = ▭ ADK

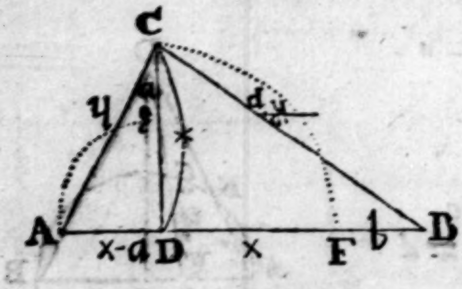
2.) By 35. III.
▭ AFB = ▭ HFG

There will come out

$$x^2 = \frac{3a^2x - b^2x - d^2x + a^2d - b^2d}{d}$$

then ſuppoſing $3a^2 - b^2 = f^2$
and $f^2 = dg$, it will be
$x^2 = gx - dx + a^2 - b^2$, &c.

Probl. 21. *In the Triangle* ABC, *demiting the perpendicular* CD *from the Angle* C, *is given the difference of both the Segments* AB, *and of the perpendicular* CD, *that is,* CE = a, *and* BF = b, *together with the proportion of the ſide* AC *to* BC, *as* a *to* d; *requir'd the ſides of the Triangle* ABC.



$CE = a = 4$
$BE = b = 3$
$d = 6$
$CD = x$
$\because AD = x - a$
$\because BD = x + b$
$AC = y$
$\therefore BC = \frac{dy}{a}$

Now 1.) In the Triangle ACD

$$y^2 = 2x^2 + a^2 - 2ax$$

2.) In the Triangle BCD

$$\frac{d^2y^2}{a^2} = 2x^2 + b^2 + 2bx$$

$$d^2y^2 = 2a^2x^2 + a^2b^2 + 2a^2bx$$

$$y^2 = \frac{2a^2x^2 + 2a^2bx + a^2b^2}{d2}$$

Whence 3.) $2x^2 + a^2 - 2ax = 2a^2x^2 + 2a^2bx + a^2b^2 \div d^2$

$$2d^2x^2 + a^2d^2 - 2d^2ax = 2a^2x^2 + 2a^2bx + a^2b^2$$

$$\begin{array}{l}2d^2\\-2a^2\end{array} x^2 = \begin{array}{l}2a^2b\\+2d^2a\end{array} x \begin{array}{l}+a^2b^2\\-a^2d^2\end{array}$$

$$d^2 = af$$

$$\begin{array}{l}2af\\-2aa\end{array} x^2 = \begin{array}{l}2a^2b\\+2d^2a\end{array} x \begin{array}{l}+a^2b^2\\-a^2d^2\end{array}$$

Divide by *a*.

$$2fx^2 - 2ax^2 = 2abx + 2d^2x + ab^2$$
$$2f - 2a = g \qquad\qquad - ad^2$$
$$x^2 = \frac{2abx + 2d^2x + ab^2 - ad^2}{g}$$
$$\frac{2ab}{g} = h \;\Big|\; \frac{b^2}{g} = l$$
$$\frac{2d^2}{g} = k \;\Big|\; \frac{d^2}{g} = m$$
$$x^2 = hx + kx + al - am$$
$$h + k = n \mid l - m = -p$$
$$x^2 = nx - ap$$

Conſtructions by Numbers.

For *af*) $a \;\ddot{-}\; d : d \;\ddot{-}\; f$
$\qquad\quad 4 \quad 6 : 6 \mid 9$

For *g*) $18 = 2f$
$\qquad\quad\;\; 8 = 2a$
$\qquad\quad\;\; 10 = g$

$$\S\; x^2 = \frac{2abx + 2d^2x + ab^2 - ad^2}{g}$$

This Equation belongs to Caſe 3. becauſe ad^2 is greater than ab^2.

$$\because x = \frac{ab + d^2}{g} \pm \sqrt{\frac{\overline{ab+d^2}\,q}{g^2} + \frac{ab^2 - ad^2}{g}}$$

$$\frac{ab+d^2}{g} = \frac{48}{10}$$

$$\left.\begin{array}{rl} \because \dfrac{\overline{ab+d^2}q}{g^2} = & \dfrac{2304}{100} \\[2ex] \dfrac{ad^2-ab^2}{g} = & \dfrac{108}{10} \\[2ex] \text{or} & \dfrac{1080}{100} \end{array}\right) \qquad \begin{array}{l} 2304 \\ \underline{1080}\ \text{ſubſtract} \\ \dfrac{1224}{100}\surd\ \dfrac{35}{10}\ \text{nearly} \\[2ex] \dfrac{48}{10}\ \text{add.} \\[2ex] \dfrac{83}{10} = x.\ q.\ e.\ f. \end{array}$$

Probl. 22. *Let there be given two quantities* = a & b, *we are to find a third* = x, *ſo that it being added to* a, *the Sum may be a Square, but the ſame being added to* b, *the Sum may be the ſide of the ſaid Square; that is*

a+x *the Square*

b+x *the ſide of the Square.*

Whence $b^2+x^2+2bx=a+x$.

Now ſince the quantity $a+x$ ought to be a Square, there will conſequently be a ſuperficies of two Dimenſions, and therefore ſhould be multiply'd into Unity as a Line. Therefore aſſuming Unity = b, it will be

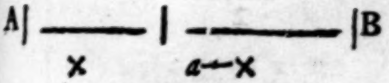
$$b^2+x^2+2bx = ab+bx$$

& $\quad x^2 = -bx+ab-b^2$

Whence $\quad x = \sqrt{\tfrac{1}{4}b^2+ab-b^2}-\tfrac{1}{2}b$

that is $\quad x = \sqrt{ab-\tfrac{3}{4}b^2}-\tfrac{1}{2}b$

Probl. 23. *The right Line* AB, $=a$, *being given, to cut it ſo that the Square of the greater part may have a given Proportion, as* a *to* f, *to the Rectangle under the whole and leſſer part.*



$$\underbrace{a^2+x^2-2ax \quad \underbrace{ax : a}} :: f,$$

$$fa^2+fx^2-2fax=a^2x$$

$$fx^2=a^2x+2fax-fa^2$$

$$a^2=df$$

$$fx^2=fdx+2fax-fa^2$$

that is, $x^2=dx+2ax-a^2$

$$d+2a=2g$$

$$x^2=2gx-a^2.$$

Probl. 24. *In the Rectangle Triangle* ABC *is given the Sum of the Sides about the right Angle, that is* AB+AC, *with the height* AD; *requir'd the Triangle.*

Let $AB+AC=a$

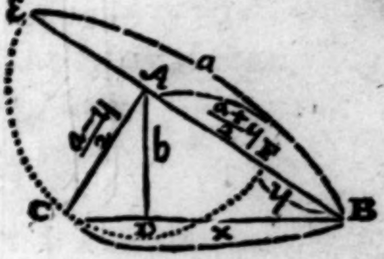
$AD=b$

$BC=x$

$BF=y$

$\because AB=\frac{a+y}{2}$

$\because AC=\frac{a-y}{2}$



Therefore 1.) In the Triangle ABC.

$$\begin{array}{ccccc} \square AB & + & \square AC & = & \square BC \\ \frac{a^2+y^2+2ay}{4} & + & \frac{a^2+y^2-2ay}{4} & = & x^2 \end{array}$$

that is, $\frac{2a^2+2y^2}{4} = x^2$ or $\frac{a^2+y^2}{2} = x^2$

whence $y^2 = 2x^2 - a^2$.

2.) By reaſon of the ſim. Triang. ABC, ADC.

$$\frac{a+y}{2} \mathrel{\overline{\cdot\cdot}} \underparen{x : \quad b} \mathrel{\overline{\cdot\cdot}} \frac{a-y}{2}$$

whence $bx = \frac{a^2 - y^2}{4}$

and $y^2 = a^2 - 4bx$

Wherefore 3.)
$$\begin{aligned} 2x^2 - a^2 &= a^2 - 4bx \\ 2x^2 &= -4bx + 2a^2 \\ x^2 &= -2bx + a^2 \\ x &= \sqrt{b^2+a^2} - b \end{aligned}$$

The ſame another way.

Let $BC = x$ | Therefore 1.) In the Triang. ABC
$CA = y$ |

$$2y^2 = x^2 - a^2 + 2ay$$
$$y^2 = ay + \frac{x^2 - a^2}{2}$$

whence $y = \sqrt{\frac{1}{4}a^2 + \frac{1}{2}x^2 - \frac{1}{2}a^2} + \frac{1}{2}a$
that is $y = \sqrt{\frac{1}{2}x^2 - \frac{1}{4}a^2} + \frac{1}{2}a$

2. By reaſon of the ſim. Triangles ABC, ADC

$$a - y \mathrel{\underset{\cdot\cdot}{-}} x : b \mathrel{\underset{\cdot\cdot}{-}} y$$

whence $bx = ay - y^2$
$$y^2 = ay - bx$$
$$y = \sqrt{\frac{1}{4}a^2 - bx} + \frac{1}{2}a$$

[Here we muſt take notice of the method of two Equations of the ſame Form; the uſe of which, as it is very fine, ſo ſometimes it is wholly neceſſary].

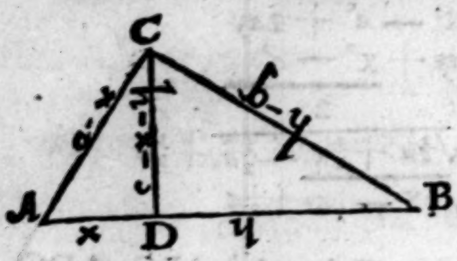
Wherefore 3.) $\sqrt{\frac{1}{2}x^2 - \frac{1}{4}a^2} + \frac{1}{2}a = \sqrt{\frac{1}{4}a^2 - bx} + \frac{1}{2}a$
and caſting away the Radical Sign,

$$\frac{1}{2}x^2 - \frac{1}{4}a^2 = \frac{1}{4}a^2 - bx$$
$$\frac{x^2}{2} = -bx + \frac{a^2}{2}$$

$x^2 = -2bx + a^2$ as above.

§. So alſo if there be given the difference of the ſaid Sides, with the height.

Probl. 25. *In this Triangle are given* $\left\{\begin{array}{l} AC+AD=a \\ BC+BD=b \\ AB+CD=c \end{array}\right.$ *it is required to find the Sides and Height ſeverally.*



Let $AD=x$

$BD=y$

$\because\ AC=a-x$

$\because\ BC=b-x$

$\because\ DC=c-x-y$

Now 1.) $\square AC-\square AD=\square BC-\square BD$

$$\begin{array}{ll} a^2+x^2-2ax & b^2+y^2-2by \\ \quad -x^2 & \quad -y^2 \end{array}$$

$$a^2-2ax=b^2-2by$$

$$2by=b^2-a^2+2ax$$

$$y=\frac{b^2-a^2+2ax}{2b}$$

2.) $\square AC-\square AD=\square CD$

$$\begin{array}{l} a^2-2ax=c^2+x^2+y^2 \\ \qquad\qquad -2cx-2cy+2xy \end{array}$$

whence $\begin{array}{l} y^2=2cy-2xy+a^2-c^2-x^2+2cx \\ \qquad\qquad\qquad\qquad\qquad\quad -2ax \end{array}$

and $y=c-x+\sqrt{a^2-2ax}$

Wherefore 3.) $c-x+\sqrt{a^2-2ax}=\dfrac{b^2-a^2+2ax}{2b}$

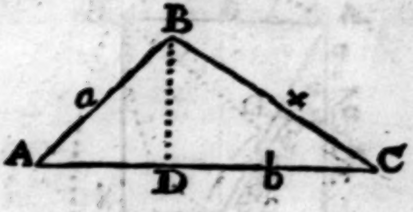
$$2bc-2bx+2b\sqrt{a^2-2ax}=b^2-a^2+2ax$$

$$\begin{array}{l} 2b\sqrt{a^2-2ax}=b^2-a^2-2bc+2ax \\ \qquad\qquad\qquad\qquad\qquad\quad +2bx \end{array}$$

b—

$$b^2 - a^2 - 2bc = -2d^2$$
$$2a + 2b = 2f$$
$$2b\sqrt{a^2 - a^2x} = 2fx - 2d^2$$
$$b^2a^2 - 2b^2ax = f^2x^2 + d^4 - 2d^2fx$$
$$f^2x^2 = 2dfx - 2b^2ax + b^2a^2 - d^4, \text{ \&c.}$$

Probl. 26. *In the Rectangle Triangle* ABC *are given* AB=a, CD=b. *Requir'd the reſt.*



1.) $x^2 - b^2 = \blacksquare BD$
2.) $a^2 - x^2 + b^2 = \blacksquare AD$
3.) $\sqrt{a^2 - x^2 + b^2} + b = \sqrt{a^2 + x_2} = AC$

Whence will come forth

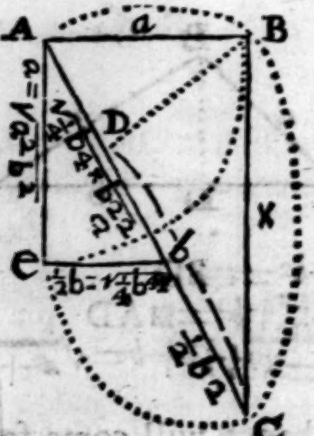
$$x^4 = b^2x^2 + b^2a^2$$
$$x^2 = \sqrt{\tfrac{1}{4}b^4 + b^2a^2} + \tfrac{1}{2}b^2$$
$$x = \sqrt{\sqrt{\tfrac{1}{4}b_4 + b^2a^2} + \tfrac{1}{2}b^2}$$

Note. This Queſtion may be ſolv'd an eaſier way, *viz.* if we ſuppoſe AD=x: But here it ought to be for an Example of an Equation of four Dimenſions.

There-

Therefore for the Conſtruction,

Taking *b* for Unity, it will be $b^2=b$, & $b^4=b$, & $\sqrt{b^2a^2}=a$, & $\sqrt{\frac{1}{4}b^4}=\frac{1}{2}b$; whence the Conſtruction is eaſie as follows.

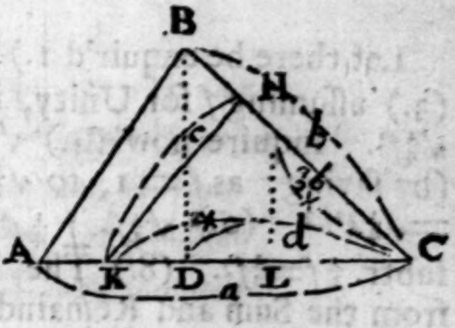


Probl. 27. *A Triangle being given* ABC *together with a right Line* KH, *to accommodate that right Line ſo within the Triangle, that it may (when it is poſſible) cut the Triangle into parts that ſhall have a given proportion.*

Or it may be propos'd thus:

In a given Triangle to apply a given right Line, that it may cut the Triangle in a given Ratio.

Let $AC = a$
$BC = b$
Line given $HK = c$
$CD = d$
$CK = x$
$\because CH \frac{ab}{2x}$



Now 1.) To find CL by the help of the ſimilar Triangles BCD, HCL.

$$b \mathbin{\cdot\!\cdot} d : \frac{ab}{2x} \;\Big|\; \frac{abd}{2bx} = CL$$

$$\because \frac{2bx^2 - abd}{2bx} = KL$$

2.) $\blacksquare HK - \blacksquare KL = \blacksquare CH - \blacksquare CL$

$$\frac{4b^2c^2x^2 - 4b^2x^4 - a^2b^2d^2 + 4ab^2dx^2 = a^2b^4 - a^2b^2d^2}{4b^2x^2}$$

$$4b^2c^2x^2 - 4b^2x^4 + 4ab^2dx^2 = a^2b^4$$

Divide by $4b^2$

$$c^2x^2 - x^4 + adx^2 = \frac{ab_2}{4}$$

$$x^4 = c^2x^2 + adx^2 - \tfrac{1}{4}a^2b2$$

$$c^2 + ad = f2$$

$$x^4 = f^2x^2 - \tfrac{1}{4}a^2b2$$

$$x^2 = \tfrac{1}{2}f^2 \pm \sqrt{\tfrac{1}{4}f4 - \tfrac{1}{4}a^2b2}$$

$$x = \sqrt{\tfrac{1}{2}f^2 \pm \sqrt{\tfrac{1}{4}f4 - \tfrac{1}{4}a^2b^2}}$$

For the Conſtruction.

Let there be enquir'd 1.) $\sqrt{ad}$ (2.) $\sqrt{ad + c^2} = f$. (3.) aſſuming f for Unity, it will be $\frac{1}{2}f = \frac{1}{2}f^2 = \sqrt{\frac{1}{4}f^4}$. Inquire now (4.) $\sqrt{\frac{1}{2}ab}$, and then (5.) $\frac{1}{2}ab$ (by ſaying: as $f = 1$, to $\sqrt{\frac{1}{2}ab}$: ſo is $\sqrt{\frac{1}{2}ab}$ to $\frac{1}{2}ab$) $= \sqrt{\frac{1}{4}a^2b^2}$. (6.) $\sqrt{\frac{1}{4}f4 \pm \frac{1}{4}a^2b2}$, which (7.) add and ſubtr. $\frac{1}{2}f = \frac{1}{2}f2$. (8.) The Root being extracted from the Sum and Remainder by the help of $f = 1$, will give both the Roots of the Equation $= x$. The greater of which is CM=CK, and the leſs CN=CH.

CLASS

CLASS III.

Of Cubic Equations, or thoſe of three Dimenſions.

Of theſe there are fourteen Caſes; which, notwithſtanding, by taking away the ſecond Term, are reduc'd to three or four, which we have mark'd in the Margin, with their proper Numbers.

$x^3 = nx^2 + apx + aaq$

$x^3 = nx^2 - apx + aaq$

$x^3 = nx^2 + apx + aaq$

$x^3 = -nx^2 - apx + aaq$

$x^3 = nx^2 + apx - aaq$

$x^3 = nx^2 - apx - aaq$

$x^3 = -nx^2 + apx - aaq$

1. $x^3 = * + apx + aaq$

2. $x^3 = * + apx - aaq$

3. $x^3 = * - apx + aaq$

$x^3 = nx^2 \; * + aaq$

$x^3 = nx^2 \; * - aaq$

$x^3 = -nx^2 \; * + aaq$

4. $x^3 = * \; * + aaq$

The Preparation of Cubic Equations and higher ones, in order to their Solution.

Which is done 1.) By reducing the Equation propos'd to the moſt ſimple Form.

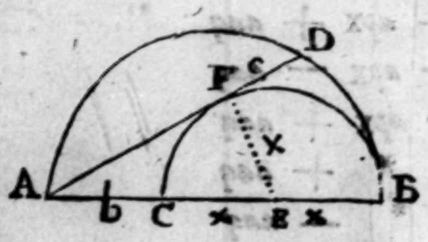
2.) By rightly diſpoſing and ordering all the Terms of the reduc'd Equation, as well thoſe which are known, as thoſe which are unknown; that is, bringing over all to one ſide, or making the whole Equation = nought.

3.) By taking away the ſecond Term of the Equation ſo diſpoſed.

4.) By applying the Equation acquir'd by this Operation to the correſpondent *Formula*, and equalling the Terms of this to the Terms of that reſpectively.

The Practiſe of what has been ſaid by Example.

Probl. 25.) In this Figure are given AC=b; FD=c; requir'd the reſt.



Suppoſing BE = x

Let it be

1.) EB to FD: as AE to AF

$$x \;\overline{\cdot\cdot}\; c : b+x \;\Big|\; \frac{bc+cx}{x}$$

2.)

2.) $\blacksquare AF + \square FE = \square AE.$

$$\frac{b^2c^2 + c^2x^2 + 2bc^2x}{x^2} + x^2 = b^2 + x^2 + 2bx$$

$$b^2c^2 + c^2x^2 + 2bc^2x = b^2x^2 + 2bx^3$$

$$2bx3 = \begin{array}{l} c^2 \\ -b^2 \end{array} x^2 + 2bc^2x + b^2c^2$$

$$x3 = \frac{\begin{array}{l} c^2 \\ -b^2 \end{array} x^2 + 2bc^2x + b^2c^2}{2b}$$

Here follows, 1. *The reduction to a ſimple Form.*

$$x3 = \frac{\begin{array}{l} -b^2 \\ +c^2 \end{array} x^2 + 2bc^2x + b^2c^2}{2b}$$

ſuppoſing $\frac{-b^2 + c^2}{2b} = -d$; & $\frac{1}{2}b = f$, it will be

$$x3 = -dx^2 + c^2x + fc^2$$

2.) *The ordering of the Terms.*

$$x3 + dx^2 - c^2x - fc^2 = 0$$

3.) *The taking away the ſecond Term.*

Where the *Rule* is thus: Suppoſe $x = y$ (or in the place of x let y be aſſum'd) + or − the known quantity of the ſecond Term, divided by the Number of the Dimenſions of the firſt, that is +, if the ſecond Term has −; and *vice verſa* − if it has +, and conſequently always with a contrary Sign; and then ſubſtitute the quantity, as follows.

§. Therefore in our Example inſtead of x we muſt take $y-\frac{1}{3}d$, ſince the ſecond Term is affected with the Sign $+$, and the number of the Dimenſions of the firſt Term is three.

Here follows the *Operation*.

$$\left.\begin{array}{l} x = y - \frac{1}{3}d \\ x^2 = y^2 - \frac{2}{3}dy + \frac{1}{9}d^2 \end{array}\right\} \text{mult.}$$

$$\left.\begin{array}{l} y^3 - \frac{2}{3}dy^2 + \frac{1}{9}d^2y \\ \quad - \frac{1}{3}dy^2 + \frac{2}{9}d^2y - \frac{1}{27}d^3 \end{array}\right\} \text{add.}$$

$$\left.\begin{array}{rl} x^3 = & y^3 - dy^2 + \frac{1}{3}d^2y - \frac{1}{27}d^3 \\ +dx^2 = & + dy^2 - \frac{2}{3}d^2y + \frac{1}{9}d^3 \\ -c^2x = & - c^2y + \frac{1}{3}dc^2 \\ -fc^2 & - fc^2 \end{array}\right\} \text{add.}$$

it will be $y^3 * - \frac{1}{3}d^2y - c^2y + \frac{2}{27}d^3 + \frac{1}{3}dc^2 - fc^2 = 0$.

or $$y^3 = * \begin{array}{l} + \frac{1}{3}d^2 \\ + c^2y \end{array} \quad \begin{array}{l} - \frac{2}{27}d^3 \\ - \frac{1}{3}dc^2 \\ + fc^2 \end{array}$$

Now ſince the laſt Term conſiſts of different Quantities, partly Negative and partly Affirmative we muſt carefully ſee which prevails, the Negative or the Affirmative, which in this place is found to be the Sign $+$.

4. *Application*

4. *Application of the formulæ.*

By taking away the ſecond term all Cubical Equations are reduc'd to the three following *formulæ.*

1.) $y^3 = * + apy + aaq$
2.) $y^3 = * + apy - aaq$
3.) $y^3 = * - apy + aaq$

Or by ſuppoſing $a = 1$.

1.) $y^3 = * + py + q$
2.) $y^3 = * + py - q$
3.) $y^3 = * - py + q$

Since therefore in the Equation found in our Example the mark + twice recurs, we may eaſily gather that it correſponds to the firſt *formula.* Wherefore it will be as follows.

	The Terms of the formulæ.		*The Terms of the Equation propos'd.*
	y^3	$=$	y^3
	$+apy$	$=$	$+\frac{1}{3}d^2y + c^2y$
	& ap	$=$	$+\frac{1}{3}d^2 + c^2$
Whence	p	$=$	$\dfrac{\frac{1}{3}d^2 + c^2}{a}$
	$+aaq$	$=$	$-\frac{2}{27}d^3 - \frac{1}{3}dc^2 + fc^2$
Whence	q	$=$	$\dfrac{+fc^2 - \frac{1}{3}dc^2 - \frac{2}{27}d^3}{aa}$

Here follows the *Conſtruction* of the *Preparation* which has hitherto been explain'd.

§. For $\frac{-b^2+c^2}{2b}$; let $\frac{c^2-b}{2b}=-d$, becauſe b^2 is greater.

§. For $\frac{\frac{1}{3}d^2+c^2}{a}$; it will be very much to the purpoſe, if $\frac{d^2}{a}=g$, & $\frac{c^2}{a}=h$; whence $h+\frac{1}{3}g=\frac{c^2+\frac{1}{3}d^2}{a}=p$.

§. For $\frac{d^2}{aa}$ (1.) as a to d, ſo is d to $\frac{d^2}{a}=g$, which was found above. (2.) As a to d, ſo is g to a fourth $=k$, $=\frac{d^3}{aa}$.

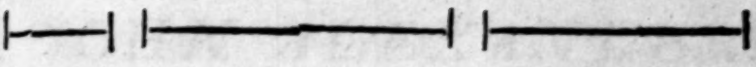
§. For $\frac{dc^2}{aa}$ (1.) As a to c, ſo is c to $\frac{c^2}{a}=h$, which was found above. (2.) As a to d, ſo is h to a fourth $=l$, $=\frac{dc^2}{aa}$.

§. For $\frac{fc^2}{aa}$ $[f=\frac{1}{2}b]$ ſince here again it is c^2, as above it will be $\frac{fc^2}{a}=fh$. (2.) As a to f, ſo is h to a fourth $=m=\frac{fc^2}{aa}$

Whence then $m^2-\frac{2}{27}k-\frac{1}{3}h=q$.

§. *Note*, For the more commodious operation, we may aſſume Lines twice as big as thoſe in the Figure above: and ſuppoſe a = to the Line here drawn.

Having perform'd the operation, the Lines neceſſary for the Solution will come forth, as here.



So much for the preparation of Cubical Equations, their Solution follows.

APPENDIX
TO THE
ALGEBRA
OF
JOHN ALEXANDER.

BY
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New Mathematical School
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LONDON:
Printed by J. BARBER, Printer to the
ſaid Hoſpital. MDCCIX.

TO THE
READER.

IN a Work of ſo ſmall a Compaſs, as the foregoing is, the Author could at leaſt but lightly touch on many things, which are very uſeful in the Practiſe of Algebra, *and open a way to the explaining the abſtruſer Parts of ſo profound an Art. I was therefore willing to ſupply ſuch a Defect, in ſome meaſure, by the following ſhort Diſcourſes with an intent to make the Book* *ſtill more uſeful to Young Beginners; and tho' this* Edition *be not compleat in every reſpect, it appears, I hope, with greater Improvements, and much more Correct.*

H. DITTON.

CHAP. I.

Of the Nature and Origin of Powers.

THO' our Author has given the Definition of Powers in their proper Place, yet I think it not amiſs to repeat it with ſome little alteration, that we may with greater eaſe underſtand the Reaſons of what follows.

A Power (properly ſo call'd) is a *Product* ariſing from *one* or *more* Multiplications, always made by the *ſame Quantity*, beginning from *Unity*, as being the firſt Term to be multiply'd. From which it is plain, that the Powers of the ſame Quantity *are a ſeries of continu'd Geometrical Proportionals.*

A Power is call'd, *Firſt*, *Second*, *Third*, *Fourth*, &c. according to the number of Involutions by which it is produc'd. And that quantity, (whether a *Number* or a *Letter*) which *ſhews* and *explains* theſe Multiplications, is commonly plac'd at the top of the Algebraical Letter, in a leſſer Character, at the right ſide of it: and *from its office*, is call'd the *Index* or *Exponent* of that quantity, to which it is ſo affix'd. By this means the Numbers 3, 5, 6, and the Letters *n*, *m*, *p*, are the Indices of the Powers, a^3, a^5, x^6, y^n, x^m, x^p, *&c.*

Hence, *vice verſa*, 'tis eaſie to underſtand what a *Root* is; by what evolutions it is ſuppos'd to be generated, and how alſo it is to be *denoted* or *expreſs'd.*

So $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{1}{4}}$, denote the *Square*, *Cube*, Biquad. Roots of the Quantity x. And $a^{\frac{2}{3}}$ ſignifies the Cubic Root of the Square of a; and $y^{\frac{3}{5}}$ the Root of the fifth Power of the Cube of y; and univerſally $x^{\frac{1}{n}}$, the Root of the n Power of x. Where notwithſtanding the Letter n may ſignifie any number, either an Integer or Fraction; and ſo $x^{\frac{1}{n}}$ may repreſent any *Perfect* or *Imperfect* Power or Root of the quantity x. And indeed the Notion of a Power is ſo general, that it may extend to all diverſity of Expreſſions whatſoever.

Nor is any diſtinction requiſite in this place, between Powers and Roots, properly ſo called; but we need only imagin that a quantity is rais'd to ſuch a Power, as its Letter or exponent Number expreſſes, whether it be an Integer or Fraction. As by $y^{\frac{n}{m}}$, is to be underſtood that the quantity y is rais'd to a Power whoſe Exponent is expreſt by $\frac{n}{m}$.

Now we are to obſerve the following Rules.

1, *Any Powers (of the ſame Quantity) are multiply'd into one another by the addition of the Exponents.*

So $a^3 \times a^5 = a^8$; $a^4 \times a^2 = a^6$; $x^{\frac{3}{2}} \times x^{\frac{5}{2}} = x^{\frac{8}{2}} = x^4$; $d^n \times d^m = d^{n \times m}$; $x^{b \times 1} + x^{c \times 1} = x^{bc \times b \times c + 2}$, *&c.*

2. *The diviſion of Powers is done by the Subſtraction of the Exponents.*

So $x^7 —: x^5 = x^2$; $x^{\frac{7}{3}} —: x^{\frac{2}{3}} = x^{\frac{5}{3}}$; $y^3 —: y^8 = y^{-5}$; $a^{\frac{3}{2}} —: a^{\frac{7}{2}} = a^{-\frac{4}{2}} = a^{-2}$; $x^m —: x^n = x^{m-n}$; $y^{\frac{m}{n}} —: y^{\frac{p}{n}} = y^{\frac{m-p}{n}}$, *&c.*

The two foregoing Rules may be alſo propos'd after this manner,

If the Indices of two Powers are added to one another, their Sum will be the Index *of the Power produc'd by their Multiplication.*

And if the Index of the dividing Power be ſubſtracted from the Index of the Power to be divided, their difference will be the Index of the Power ariſing from the Diviſion.

3. *The elevation of Powers to other Powers is done by multiplying the Exponents.*

Or thus. If any given Power is to be elevated to another given Power, the Index of the Power propos'd muſt be multiply'd into the Index of that Power, to which the firſt ought to aſcend.

So x^5 elevated to the third Power, is x^{5+3}, or x^{15}; to the ſecond Power is $x^{5 \times 2} = x^{10}$; to the ſeventh $x^{5 \times 7} = x^{35}$; y^2 to the Power n, is y^{2n}; y^p to the Power q, is y^{pq}, *&c.*

4. *The extraction of all manner of Roots is done by dividing the Exponents.*

Or otherwiſe. If any Root is to be extracted, the Index of the Power propos'd, is to be divided by the Index of that Power, whoſe Root is requir'd.

So the Square Root from a^8, is $a^{8-:2} = a^4$; The Cube Root is $a^{8-:3}$; or $a^{\frac{8}{3}}$; the Bi-quad; is $a^{8-:4}$, that is a^2. So the Root $\frac{2}{5}$ from y^4, is $y^{\frac{2}{2}2} = y^{10}$; the Root m from a^n, is $a^{\frac{n}{m}}$; the Root n from x^{2mn-nn}, is x^{2m-n}, &c.

Yet we muſt take notice that both the elevation of Powers, and the extraction of Roots, may be perform'd by bare Multiplication.

So that theſe two are very well reduc'd to one and the ſame kind of Operation. For as every Root may be conſider'd as a kind of a Power, ſo the extraction of a Root may be look'd upon as the elevation of a quantity to a certain Power.

By this means the Cube Root from a^4, is $a^{4-:3}$, or $a^{4\times\frac{1}{3}}$; and the Root n, from x^{3m}, is either $x^{3m-:n}$, or $x^{3m\times\frac{1}{n}}$, &c.

For it comes to the ſame, to divide one quantity by another, and to multiply by the *Reciprocal* of the Diviſor.

And from hence appears the reaſon of the common operations with *Logarithms*. For *Logarithms* are the Exponents of Numbers or *Ratio's*. Hence therefore to perform *Multiplication* by the Log. is to add the Log. of the Numbers to be multiply'd. *Diviſion* is perform'd by ſubſtracting the Log.

Logarithm of the *Diviſor* from that of the *Dividend*. The Elevation of Numbers to Powers is done by multiplying the Logarithm of the Number, whoſe Power is to be rais'd by the *Exponent* of that Power it is to be rais'd to. On the other hand, Extraction of Roots by Diviſion of the Logarithm by the ſaid Exponent.

In compound Quantities the ſame Laws obtain which we have ſhewn hitherto in the ſimple ones.

The Reaſons of all which will be very perſpicuous (to a diligent Obſerver) from the *Geneſis of Powers*: Of which it is ſufficient to give a Hint.

CHAP.

CHAP. II.

OF

Surd or *Irrational Quantities.*

THere are certain Roots of Rational Numbers, which cannot be expreſs'd in *Rational Numbers*; and ſuch are call'd *Surd* or *Irrational.* So that by *Surd* Quantities, we mean nothing elſe than Roots of that kind and Nature, which are not capable of expreſſion in Rational Numbers. And what is ſaid of *Numbers*, is to be underſtood alſo of Quantities deſign'd by *Algebraic Species*.

The common Mark of Radicality is this $\surd$; joining to it either a Number or Letter, which declares the Nature of the Root, ſo $\surd(3)\,120$, or $\sqrt[3]{120}$, denotes the Cube-Root of 120: $\surd(4)\,aab$ or $\sqrt[4]{aab}$, the Biquad. of the Expreſſion aab: and $\sqrt{aaa+abb}$ the Square-Root of the Quantity $aaa+abb$. For ſometimes there are ſeveral Terms included under one and the ſame Radical Mark, and ſuch Roots are call'd *Univerſals*, ſo $\sqrt[3]{ccd-n^3+m^3}$, ſignifies the Cube-Root of the Difference between $ccd+m^3$ & n^3, & $\sqrt[3]{p^3+\sqrt{a^3b^3+c^4d^2}}$ the Cube-Root of the quantity p^3, added to the Square-Root of the quantity

$a^3b^3+c^4d^2$; & $\sqrt[4]{q^4}+\sqrt[3]{a^3p^3+n^3m^3-bb\sqrt{aa-cc}}$. Bi-quadratic Root from q^4, added to the Cube Root, of the difference which is between $a^3p^3+n^3m^3$, and bb multiply'd into the Square Root of the quantity $aa-cc$. And by this Rule others may be judg'd of.

Surd Quantities, if they have different Radical Signs, are very eaſily reduc'd to a common Sign, by a vulgar operation with the *Exponents*, viz: by bringing them to a *common Denominator*.

So $\sqrt[3]{5}$ & $\sqrt[4]{7}$, that is $5^{\frac{1}{3}}$ & $7^{\frac{1}{4}}$, if the Fractions are reduc'd, they are $5^{\frac{4}{12}}$, & $7^{\frac{3}{12}}$ or $\sqrt[12]{5^4}$, & $\sqrt[12]{7^3}$, or $\sqrt{}$ (12) 625, & $\sqrt{}$ (12) 343. After the ſame manner, $\sqrt[2]{aq}$, & $\sqrt[3]{aaq}$, that is $aq^{\frac{1}{2}}$ & $aaq^{\frac{1}{3}}$, are reduc'd to $aq^{\frac{3}{6}}$ & $aaq^{\frac{2}{6}}$, or $\sqrt[6]{aq^3}$, & $\sqrt[6]{aaq^2}$, or $\sqrt{}$ (6) a^3q^3, & $\sqrt{}$ (6) a^4q^2. But to avoid miſtakes we are to remember, that the Exponent always affects the whole Expreſſion, and not the laſt Letter only (which it immediately follows). As for Example, by the Expreſſion $\sqrt[6]{aaq^2}$, we mean the Root of the ſixth Power, not of $aaqq$, but of the Square of aaq, that is a^4q^2. And the like Conſtruction is to be made in the reſt. We will give but one Example, which may ſerve as a Rule for every Numerical Operation. Let there be propos'd $\sqrt[m]{a^q}$, & $\sqrt[n]{ep}$, or $a^{\frac{q}{m}}$ & $e^{\frac{p}{n}}$, which by reduction are $a^{\frac{qn}{mn}}$ & $e^{\frac{mp}{mn}}$. Then expounding the Fractions $\frac{qn}{mn}$, & $\frac{pm}{mn}$, in the *leaſt* Numbers, which have

a

a *common Denominator*, let them be $\frac{r}{d}$ & $\frac{s}{d}$ reſpectively. Therefore $a^{\frac{r}{d}}$ & $e^{\frac{s}{d}}$ or $\sqrt[d]{a^r}$ and $\sqrt[d]{e^s}$, will be the Expreſſions ſought, which have a *common* and the *leaſt* radical Sign. *For example*, if $\frac{q}{m}=\frac{1}{2}$, & $\frac{p}{n}=\frac{3}{4}$ if then $\frac{qn}{mn}=\frac{4}{8}$, and $\frac{pm}{mn}=\frac{6}{8}$, as $a^{\frac{qn}{mn}}$, and $e^{\frac{pm}{mn}}$ are $a^{\frac{4}{8}}$ and $e^{\frac{6}{8}}$, or $\sqrt[8]{a^4}$ and $\sqrt[8]{e^6}$ that is (if they be brought to the loweſt Terms) $\sqrt[4]{a^2}$ and $\sqrt[4]{e^3}$, ſo that d in this Example is 4.

A Rational Quantity is reduc'd to any Radical Sign by *Involution* (or Multiplication into its ſelf) agreeable to the nature of that Sign.

So the Number 5 is reduc'd to the Sign $\sqrt[3]{\ }$ by Cubic Multiplication, and is $\sqrt[3]{125}$; and $e+x$ to $\sqrt[2]{\ }$, by ſquaring it; and then inſtead of it is written $\sqrt[2]{e^2+2ex+xx}$; or if to $\sqrt[3]{\ }$, there will ariſe $\sqrt[3]{e^3+3e^2x+3ex^2+x^3}$. And ſo of the reſt. It muſt not be therefore taken for granted, that an Expreſſion which is included under any Radical Sign, is always *Surd*, or of ſuch a kind as cannot be expreſt by a rational Quantity. Since Rational Quantities themſelves are reduc'd to ſuch a Form and often] lie conceal'd under this diſguiſe of Radicality. Thus, $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt[4]{256}$, $\sqrt[5]{3125}$, *&c.* are the natural Numbers, 2, 3, 4, 5, *&c.* as is manifeſt from the Evolution of each.

Surd Roots are often reduc'd to *more ſimple ones*, by *dividing* the Quantities comprehended under the Radical Sign, by *ſome Power* from which they were produc'd by Multiplication.

So $\sqrt{245xx}$, is reduc'd to $7x\sqrt{5}$, becauſe $245xx = 49xx$ in 5, whoſe Square Roots are 7x & $\sqrt{5}$. In the ſame manner $\sqrt[3]{72e^3}$ comes forth $2e\sqrt[3]{9}$, for $72e^3$, is produc'd by multiplying $8e^3$ into 9, whoſe Cube Roots are $2e$, & $\sqrt[3]{9}$. In like manner for $\sqrt{x^3b + x^2b^2}$, I write $x\sqrt{xb + bb}$, becauſe making a diviſion of it $x^3b + x^2b^2$, by xx, ariſes $xb + bb$. And inſtead of this following, $\sqrt{ad^2 + 2adx + axx + 2bd^2 + 4dbx + 2bx^2}$, we may ſubſtitute $d + x\sqrt{a + 2b}$, for the firſt Expreſſion may be divided by the Square $d^2 + 2dx + x^2$, and the Quotient is $a + 2b$.

So for the ſame reaſon, inſtead of $\sqrt{\frac{a^2n^2q^2}{r^2x^2} + \frac{8n^2q^2d}{rrx}}$ may be written, $\frac{nq}{rx}\sqrt{aa + 8dx}$.

But if amongſt the Diviſors of the Surd Quantity propos'd, ſuch a Power cannot be found, by which the Diviſion is to be perform'd, then, by this *method*, it cannot be reduc'd to a more ſimple Form. But yet, if that be deſir'd, it may be after ſome manner expreſt in the form of a Fraction. *For Example*, Let $\sqrt{10aa}$; be propos'd for an equivalent Fraction, write $2a\sqrt{n}$, or $3a\sqrt{n}$, or $5a\sqrt{n}$, or $10a\sqrt{n}$, or *a* multiply'd into any other Number taken at Pleaſure. Then if $\sqrt{10aa} = 2a\sqrt{n}$, it will be $10aa = 4aan$, & $n = \frac{10}{4} = \frac{5}{2}$; ſo that $\sqrt{10aa} = 2a\sqrt{\frac{5}{2}}$ if $\sqrt{10aa} = 5a\sqrt{n}$, then $10aa = 25aan$, & $n = \frac{10}{25} = \frac{2}{5}$, & $\sqrt{10aa} = 5a\sqrt{\frac{2}{5}}$. And ſo may be expreſt after an infinite Number of ways.

But we are further to take notice, that of Surd Quantities ſome are *Commenſurable*; that is, are in relation to one another, as a *Rational Quantity* to a *Rational Quantity*: and others *Incommenſurable*.

And it will eaſily appear, how any given Surd bears itſelf to another, as to theſe reſpects, by reducing both of them to the moſt ſimple Terms. Let the *greateſt common Diviſor of both* be found, and conſider the *Quotients*, whether they be to one another, as a rational Quantity to a rational Quantity. For, when this comes to paſs, they are call'd Commenſurable; otherwiſe Incommenſurable. Thus dividing $\sqrt{75xx}$ & $\sqrt{27xx}$, by $\sqrt{3}$, their greateſt common Diviſor, then come forth $\sqrt{25xx}$ & $\sqrt{9xx}$, or $5x$ & $3x$: and there the Surds propos'd are to one another, as Number 5 to 3, and for that reaſon *Commenſurable*. So $\sqrt{q^4+qqbb}$ & $\sqrt{qqbb+b^4}$, divided by $\sqrt{qq+bb}$; give $\sqrt{qq}$ & $\sqrt{bb}$, or q & b, ſo that the firſt $= q\sqrt{qq+bb}$, and the ſecond $= b\sqrt{qq+bb}$, and therefore they are to one another, as the Rational q to the Rational b. After the ſame manner, $\sqrt{\frac{eexx}{bb}+\frac{4nqxx}{bb}}$ & $\sqrt{\frac{bbeenn}{qqxx}+\frac{4bn^3}{qxx}}$ divided by $\sqrt{ee+4nq}$, where come forth $\frac{x}{b}$, & $\frac{bn}{qx}$. And alſo $\sqrt{\frac{9a^2b-12abx+4bx^2 9a^2x+12ax^2-4x^3}{bb}}$ & $\sqrt{\frac{16bc^2-24bcx+9bx^2-16c^2x+24cx^2-9x^3}{dd}}$ divided by $\sqrt{b-x}$ make $\sqrt{\frac{9a^2-12ax+4x^2}{bb}}$, & $\sqrt{\frac{16c^2-24cx+9x^2}{dd}}$ or $\frac{3a-2x}{b}$; & $\frac{4c-3x}{d}$.

So that the two *firſt* are $\frac{x}{b}\sqrt{ee+4nq}$, & $\frac{bn}{qx}\sqrt{ee+4qn}$ and therefore, as $\frac{x}{b}$ to $\frac{bn}{qx}$ or as qxx to bbn: the two latter are, $\frac{3a-2x}{b}\sqrt{b-x}$, & $\frac{4c-3x}{d}\sqrt{b-x}$; which bear the ſame proportion as $\frac{3a-2x}{b}$ to $\frac{4c-3x}{d}$, or $3ad-2dx$ to $4bc-3bx$.

Hence will it be very eaſy to *Add* and *Subſtract* Surd Quantities. For, reducing them to the moſt ſimple Terms, there is nothing more to do, if they are found *Commenſurable*, than that the *Quantities plac'd without the Radical Sign*, be *added* to, or *ſubſtracted* from one another, and that the *ſum* or *difference* be multiply'd into the Quantity included *under the Sign.*

As $\sqrt{75aa}$ & $\sqrt{27aa}$, or $5a\sqrt{3}$ & $3a\sqrt{3}$; if they be *Added*, they make $8a\sqrt{3}$; but if the leſſer be *Subſtracted* from the greater, there remains $2a\sqrt{3}$. So $\sqrt{a^4+aabb}$ & $\sqrt{aabb+b^4}$, or $a\sqrt{aa+bb}$, & $b\sqrt{aa+bb}$ by *Adding* and *Subſtracting*, make the Sum $a+b\sqrt{aa+bb}$ and the *difference*, $a-b\sqrt{aa+bb}$, or $b-a\sqrt{aa+bb}$, according as a or b is greater.

In like manner $\sqrt{\frac{4eedd-4eezz}{cc}}$ (that is by reduction, $\frac{2e}{c}$, $\sqrt{dd-zz}$) & $\sqrt{dd-zz}$, or Ix $\sqrt{dd-zz}$; give the *Sum* $\frac{2e+c}{c}\sqrt{dd-zz}$, and the *difference* $\frac{\overline{2e-c}}{c}\sqrt{dd-zz}$ or $\frac{\overline{c-2e}}{c}\sqrt{dd-zz}$.

But where the Surd Quantities are *Incommenſurable*, they are *Added* and *Subſtracted* by the means of the Signs + and —, by which they ought to be joyn'd together. As the Sum of the two Quantities $e+z$ & $\sqrt{ee+zz}$, is $e+z+\sqrt{ee+zz}$. The *Difference* (if the laſt be ſubſtracted from the firſt) $e+z-\sqrt{ee+zz}$; nor will it be worth while to produce more Examples of this kind, becauſe the Operation is perform'd after the ſame manner in other Caſes, having firſt found the Quantities propos'd to be Incommenſurable.

The *Multiplication* of (commenſurable) Surd Quantities is perform'd by multiplying the Parts which are *without the Radical Sign*, into one another; and the Product into that which is contain'd *under the Radical Sign*. So multiplying $\sqrt{20xx}$ by $\sqrt{125xx}$, or $2x\sqrt{5}$, by $5x\sqrt{5}$, is made $10xx \times$ by $5=50xx$; & $\sqrt{16eed+16eec}$ by $\sqrt{81eed+81eec}$, is made $4e\sqrt{d+c}$ by $9e\sqrt{d+c}$ there riſes $36ee\,\overline{d+c}=36eed+36eec$; & $\sqrt{q^4+qqbb}$ by $\sqrt{qqbb+b^4}$ or $q\sqrt{qq+bb}$ by $b\sqrt{qq+bb}$, is $bq \times \overline{qq+bb}=bq^3+qb^3$. Likewiſe, if the Expreſſion $\sqrt{z^4+6z^3+21zz+72z+108}$, be propos'd to be multiply'd by the Expreſſion $\sqrt{z^4-10z^3+37zz-120z+300}$, that is, $\overline{z+3}\sqrt{zz+12}$, by $\overline{5-z}\sqrt{zz+12}$: then multiplying $z+3$ & $5-z$, there comes $15+2z-zz$, which being multiply'd by $zz+12$, gives $180+24z+3zz+2z^3-z^4$, for the Product Sought.

But if the Surds given be *Incommenſurable*, we need only multiply the Quantities *comprehended under the Radical Signs*, into one another; and prefix the common Radical Sign to the Product.

So $\sqrt{5}$ by $\sqrt{7}$, is $\sqrt{35}$; $\sqrt[3]{4}$ by $\sqrt[3]{9}$, is $\sqrt[3]{36}$; $\sqrt{ax}$ by $\sqrt{dz}$, is $\sqrt{adxz}$; $\sqrt{cc+dd}$ by $\sqrt{cc-dd}$, gives $\sqrt{c^4-d^4}$: In like manner, $e+\sqrt{bx}$, multiply'd into it ſelf, makes $ee+bx+2e\sqrt{bx}$: that is, the Sum of the Squares, *Rational*, and for the double Rectangle, the *Irrational* Quantity $2e\sqrt{bx}$ or $\sqrt{4eebx}$. After the ſame manner if $\sqrt{pp+xx}+\sqrt{pp-xx}$ were to be multiply'd by $\sqrt{pp+xx}-\sqrt{pp-xx}$; then multiplying the two firſt Members into themſelves, is produc'd $pp+xx$; and alſo the two laſt (having a regard to the contrary Signs) into one another, is produc'd $-pp+xx$; and by multiplying the firſt and laſt into themſelves, that is, $\sqrt{pp+xx}$ into $\sqrt{pp-xx}$, $\sqrt{pp+xx}$ into $-\sqrt{pp-xx}$, the Products having contrary Signs, deſtroy one another, ſo that there remains only $pp+xx$ and $-pp+xx$, to be added to one another for the Product, which is therefore only equal to $2xx$.

Moreover, the *Diviſion* of Surd Quantities, if they be *commenſurable*, is perform'd by dividing the Quantities which are without the Radical Sign; for that which riſes from the Diviſion is the *Quotient*.

So to divide $\sqrt{e^8+e^6dd}$ by $\sqrt{e^4a^4+e^2a^4d^2}$, or $e^2\sqrt{e^4+eedd}$, by $a^2\sqrt{e^4+e^2d^2}$, is $\frac{e^2}{a^2}$, and $\sqrt[3]{abc^3}$ divided by $\sqrt[3]{ab}$, that is $c\sqrt[3]{ab}$ by $\sqrt[3]{ab}$ gives c. And in like manner we are to do with the reſt.

But if they are *Incommenſurable*, then the Quantities only are to be divided *which are comprehended under the Radical Signs*, and the *common* Radical Sign to be prefix'd to the reſult.

As $\sqrt[3]{a^3b-ab^3}$ divided by $\sqrt[3]{aa-bb}$, makes $\sqrt[3]{ab}$; for a^3b-ab^3 divided by $aa-bb$ makes ab the Quotient.

And indeed if the Radical Signs are *different*, then the given Quantities are to be reduc'd to the *common* Radical *Sign*: Which is alſo to be obſerv'd in every other Operation with Heterogeneal Surds.

So to divide x^3+xcc by $\sqrt{x^4+xxcc}$ reducing the expreſſion x^3+xc^2, to a Surd of the Quadratic Order, there comes out $\sqrt{x^6+2x^4c^2+x^2c^4}$; and dividing this by the former, ariſes $\sqrt{xx+cc}$. Further, if $x^3+xcc+xc\sqrt{xx+cc}$ is to be divided by $x\sqrt{xx+cc}$; then the firſt Member x^3+xcc, or $\sqrt{x^6+2x^4c^2+x^2c^4}$ divided by $x\sqrt{xx+cc}$, makes in the Quotient $\sqrt{xx+cc}$; and the ſecond $xc\sqrt{xx+cc}$ divided alſo by the ſame $x\sqrt{xx+cc}$ gives c; whence the Quotient ſought will be $\sqrt{xx+cc}+c$.

But in *Univerſal* Surds, where the Expreſſions given are ſo perplex'd and intricate, or of ſuch a nature, that the Diviſion will not proceed accurately and without a Remainder, then the Quotient may be expreſt after the manner of a Fraction, by placing the dividing Quantity, as a Denominator, under the others, and *prefixing the Univerſal Radical Sign*. Thus, if we were to expreſs the Quotient of this Diviſion, *viz.* $\sqrt{\sqrt{abb+bcd}}$ by $\sqrt{\sqrt{ac-dd}}$ we may write, $\sqrt{\dfrac{\sqrt{abb+bcd}}{\sqrt{ac-dd}}}$; and

$\sqrt{\frac{\sqrt{5}+\sqrt{8}}{\sqrt{7}-\sqrt{2}}}$, for the Quotient which will ariſe from the Diviſion of $\sqrt[3]{\sqrt{5}+\sqrt{8}}$ by $\sqrt[3]{\sqrt{7}-\sqrt{2}}$. And according to theſe Examples, which we have hitherto briefly propos'd, may the young Beginner in *Algebra* form others to himſelf; ſo that there is no occaſion to handle theſe things more at large.

It remains only for us to ſhew in a few words, a certain very uſeful and eaſie Practiſe, relating to *Multinomials*, compounded of Irrational Quantities. That is, how Rational Quantities may be produc'd, by proper and (as often as there is occaſion) by repeated multiplication of theſe Polynomials.

And that we may come to a clearer knowledge of the true and genuine Foundation of this Operation, we ſhall in this place expreſs Surd Quantities, not by their Radical Signs, but by their Indices (or Exponents).

For by this means the Practiſe is much more eaſie and ſimple, and the Products ariſing in a multitude of Caſes are foreſeen, and form'd with leſs labour.

For Example. Let there be propos'd $\sqrt{a}+\sqrt{b}$, or $a^{\frac{1}{2}}+b^{\frac{1}{2}}$; from the Multiplication of which, or by ſome other Quantity, ſhould ariſe a *Rational Quantity*. Firſt, it is manifeſt, That the Quantity which muſt anſwer this purpoſe, ought to conſiſt of as many *Names* as the Quantity propos'd does: that is, it ought to be a *Binomial*. Then if $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ be multiply'd into it ſelf, that is, if we multiply $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ the Product will comprehend two Rational Members, and a third Irrational, or Surd, *viz.* a, b, and $2a^{\frac{1}{2}}b^{\frac{1}{2}}$, or $2\sqrt{a}\times\sqrt{b}$, that is, $2\sqrt{ab}$. Wherefore $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ is not that proper Binomial, which being multiplied into the propos'd Surd, will produce a Rational Quantity.

Take therefore the correſpondent *Reſidual*, *viz.* $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ (which is alſo a *Binomial* in another ſenſe, tho' by reaſon of its Negative Sign, it is diſtinguiſh'd from a *Binomial properly ſo called*) then by multiplying $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ we have the Rational Product $a-b$: for the Terms in which $a^{\frac{1}{2}}$ and $b^{\frac{1}{2}}$ multiply'd into themſelves are found, by reaſon of their contrary Signs deſtroy one another.

$$\begin{array}{ll} \text{Mult.} & a^{\frac{1}{2}}+b^{\frac{1}{2}} \\ \text{By} & a^{\frac{1}{2}}-b^{\frac{1}{2}} \\ \hline \text{Prod.} & a+a^{\frac{1}{2}}b^{\frac{1}{2}}-a^{\frac{1}{2}}b^{\frac{1}{2}}-b, \text{ or } a-b. \end{array}$$

So it is in Binomials, which conſiſt of *pure Quadratic Roots.* In others more Multiplications are required.

Let there be a Binomial, conſiſting of Biquadratic Roots: *For example,* $\sqrt[4]{a}+\sqrt[4]{b}$, or $a^{\frac{1}{4}}+b^{\frac{1}{4}}$ then as before.

Mult. $a^{\frac{1}{4}} + b^{\frac{1}{4}}$

By $\underline{a^{\frac{1}{4}} - b^{\frac{1}{4}}}$

Prod. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ or $\sqrt{a} - \sqrt{b}$.

Now Mult. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$

By $\underline{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$

Prod. $a - b$, as before.

Let there be propos'd a Binomial conſiſting of the Squared Roots of the Biquadratics.

Mult. $a^{\frac{1}{8}} + b^{\frac{1}{8}}$

By $\underline{a^{\frac{1}{8}} - b^{\frac{1}{8}}}$

Prod. $a^{\frac{1}{4}} - b^{\frac{1}{4}}$, or $\sqrt[4]{a} - \sqrt[4]{b}$.

Again, Mult. $a^{\frac{1}{4}} - b^{\frac{1}{4}}$

By $\underline{a^{\frac{1}{4}} + b^{\frac{1}{4}}}$

Prod. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$, or $\sqrt{a} - \sqrt{b}$.

Laſtly, Mult. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$

By $\underline{a^{\frac{1}{2}} + b^{\frac{1}{2}}}$

Prod. $a - b$

And thus we may go on *ad infinitum*, when the Roots belong to the *Quadratic Order*. In other Roots it will not ſucceed, ſince there will ariſe partly Surd, and partly Rational Products; and that from the Nature of the Exponents. As for Example.

Mult. $a^{\frac{1}{3}}+b^{\frac{1}{3}}$, that is $\sqrt[3]{}a+\sqrt[3]{}b$

By $a^{\frac{1}{3}}-b^{\frac{2}{3}}$, that is $\sqrt[3]{}a-\sqrt[3]{}b$,

Prod. $a^{\frac{2}{3}}-b^{\frac{2}{3}}$, or $\sqrt[3]{}aa-\sqrt[3]{}bb$,

If you multiply the latter Product, *viz.* $a^{\frac{2}{3}}-b^{\frac{1}{3}}$ into $a^{\frac{2}{3}}+b^{\frac{2}{3}}$, you will only have $a^{\frac{4}{3}}-b^{\frac{4}{3}}$, and if you proceed the ſame way, *ad infinitum*, you will never come at a rational Product. But if $a^{\frac{1}{3}}-b^{\frac{2}{3}}$ be multiply'd into $a^{\frac{1}{3}}+b^{\frac{1}{3}}$ there will ariſe this mix'd Product $a-b+a^{\frac{2}{3}}\ b^{\frac{1}{3}}-a^{\frac{1}{3}}\ b^{\frac{2}{3}}$, that is $a-b+\sqrt[3]{}aa\sqrt[3]{}b-\sqrt[3]{}a\sqrt[3]{}bb$.

In *Trinomials*, or *Quadrinomials*, compos'd of pure *Quadratic Roots*, the Multiplication is to be made by proper Trinomials or Quadrinomials; by which means *Quadrinomials* will be brought down to *Trinomials*, and *Trinomials* to *Binomials*, and by Multiplication, with their proper *Reſiduals*, down to *Rational Quantities.*

Suppoſe *for Example.* The Trinomial $\sqrt{}a+\sqrt{}b+\sqrt{}c$, or $a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}$, if this ſhould be multiply'd with a Trinomial conſiſting of the ſame *Names*, and affected with the ſame Sign, it is manifeſt, that, beſides the rational Terms $a+b+c$, ſix other irrational ones would alſo remain in the Product; ſo that no good could come of this operation,

operation. But if changing the Sign of one of the *Names*, we ſhould make uſe of the Multiplier $\sqrt{a}+\sqrt{b}-\sqrt{c}$, then ſo many Members would deſtroy one another in the Product, that the Sum of the remainders would not be more than a Binomial.

Mult. $a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}$
By $a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{\frac{1}{2}}$

Prod. $a+b-c+2a^{\frac{1}{2}}b^{\frac{1}{2}}$, or ſuppoſing $a+b-c=d$, it will be $d+2\sqrt{ab}$. Then taking the remainder $d-2\sqrt{ab}$, or $2\sqrt{ab}-d$.

Mult. $d+2a^{\frac{1}{2}}b^{\frac{1}{2}}$
By $d-2a^{\frac{1}{2}}b^{\frac{1}{2}}$

Prod. Rat. d^2-4ab, or $4ab-d$ if $4ab>d$.

Now as in a Trinomial, the Sign of one Name only is to be chang'd; ſo in a Quadrinomial, *the Signs of two Names are to be chang'd*, that by Multiplication it may be brought down to a Trinomial.

Suppoſe $a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}+d^{\frac{1}{2}}$
Mult. by $a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{\frac{1}{2}}-d^{\frac{1}{2}}$ & rejecting contraries.

The Prod. will be $a+b-c-d+2\sqrt{ab}-2\sqrt{dc}$, and in the place of $a+b-c-d$ writing f, we have the Trinomial $f+2\sqrt{ab}-\sqrt{dc}$. And working as before.

Mult. $f+2a^{\frac{1}{2}}\ b^{\frac{1}{2}}-2d^{\frac{1}{2}}\ c^{\frac{1}{2}}$

By $f+2a^{\frac{1}{2}}\ b^{\frac{1}{2}}+2d^{\frac{1}{2}}\ c^{\frac{1}{2}}$, & rejecting contraries

The Prod. is $f^2+4ab-4dc+4fa^{\frac{1}{2}}\ b^{\frac{1}{2}}$; and ſuppoſing $f^2-4ab-4dc=n^2$, the Product is n^2+4f $a^{\frac{1}{2}}\ b^{\frac{1}{2}}$.

Laſtly, Mult. $n^2+4fa^{\frac{1}{2}}\ b^{\frac{1}{2}}$

by $n^2-4fa^{\frac{1}{2}}\ b^{\frac{1}{2}}$

There ariſes the Prod. n^4-16f^2ab, of which no part is irrational.

The uſe of this Practice is in *dividing Surd Quan ties*, that is, when the Diviſor it ſelf is either a Binomial or Trinomial, *&c.* For reducing the Diviſor to a ſimple rational Quantity, and multiplying the Quantity to be divided, by all the ſame Quantities which we made uſe of in finding out the rational Diviſor; we ſhall obtain by this means a new Diviſor and a new Dividend, which will have the ſame proportion to one another as the former. As if there be requir'd the Quotient of the Diviſion of $\surd 8$ by $\surd 3+\surd 2+1$; I multiply the Diviſor by $\surd 3+\surd 2-1$, and I find $\surd 24+4$, which being multiply'd into $\surd 24-4$, (it's proper *Reſidual*) gives the rational Diviſor 8. And becauſe the Diviſor $\surd 3+\surd 2+1$, was firſt multiply'd by $\surd 3+\surd 2-1$, and the Product by $\surd 24-4$, therefore both the Quantity *to be divided*, muſt be multiply'd by $\surd 3+\surd 2-1$, and the Product by $\surd 24-4$, ſo that we may have $8+\surd 12$

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-192 ſor a new Dividend, which being divided by 8, there comes out the Quotient $1+\sqrt{2}-\sqrt{3}$.

And this may ſuffice to be ſaid concerning Surd Quantities.

I ſhall now only add an excellent and eaſie Rule, for the extracting of Roots out of Binomials, conſiſting of Integer and Radical Quadratics, which that extraordinary Gentleman Sir *Iſaac Newton* has ſupply'd us with, in the late Edition of his *Algebra*. Let A be the greater part of any Binomial, and B the leſſer: it will then be $\frac{A+\sqrt{AA-BB}}{2}$ the Square of the greater part of the Root, and $\frac{A-\sqrt{AA-BB}}{2}$ the Square of the leſſer part, which is to be joyn'd to the greater with the Sign of B: and therefore that will be $\sqrt{\frac{A+\sqrt{AA-BB}}{2}}$ and $\sqrt{\frac{A-\sqrt{AA-BB}}{2}}$ the very parts which compoſe the Root. *For inſtance.* Let there be propos'd a Binomial $14+\sqrt{180}$, where $A=14$, $B=\sqrt{180}$, for it is $14\times14\,(=196)>180$. Now $AA-BB=16$, $\therefore \sqrt{AA-BB}=4$, $\because \frac{A+\sqrt{AA-BB}}{2}=\frac{14+4}{2}=9$. In like manner, $\frac{A-\sqrt{AA-BB}}{2}=\frac{14-4}{2}=5$. So that the Squares of the Roots are 9 and 5, whence the Root it ſelf $=3+\sqrt{5}$.

Suppoſe the Binomial $\frac{2a^4+x^4}{x^2}+\frac{2a^2}{x}\sqrt{\frac{a^4}{x^2}+x^2}$ whoſe Root is required. Here $\frac{2a^4+x^4}{x^2}$, or $\frac{2a^4}{x^2}+x^2=$ A, and $\frac{2a^2}{x}\sqrt{\frac{a^4}{x^2}+x^2}=$ B, and $AA-BB=x^4$ $\therefore \sqrt{AA-BB}=x^2$. Therefore it will be $\frac{A+\sqrt{AA-BB}}{2}=\frac{a^4}{x^2}+x^2$, and $\frac{A-\sqrt{AA-BB}}{2}=\frac{a^4}{x^2}$. Whence the Root it ſelf $=\frac{a^2}{x^2}+\sqrt{\frac{a^4}{x^2}+xx}$.

Again, let there be propos'd $aa+5ax-2a\sqrt{ax+4xx}$ (which is one of the Examples of the foreſaid *Great Man*) ſuppoſing $aa+5ax$ in the place of A, and $2a\sqrt{ax+4xx}$ for B, it will be $AA=a^4+10a^3x+25a^2x^2$, $BB=4a^3x+16aaxx$, $AA-BB=a^4+6a^3x+9a^2x^2$, $\sqrt{AA-BB}=aa+3ax$: whence $\frac{A+\sqrt{AA-BB}}{2}=\frac{aa+5ax+aa+3ax}{2}=aa+4ax=$ the Square of the greater part of the Root, and $\frac{A-\sqrt{AA-BB}}{2}=\frac{aa+5ax-aa-3ax}{2}=ax$. And therefore the whole Root $=\sqrt{aa+4ax}$, $-\sqrt{ax}$, for the leſſer part is to be joyn'd to the greater, with the Sign of B, which in this place is Negative.

Now the truth of this Rule is *demonſtrated* from hence, that ſince the greater part of the Root, according to its direction, is $\sqrt{\frac{A+\sqrt{AA-BB}}{2}}$ and the leſſer $\sqrt{\frac{A-\sqrt{AA-BB}}{2}}$, and therefore their Sum equal to the whole Root: If this Sum be multiply'd into it ſelf, the Product ought to be equal to A+B. And from the conſtitution of theſe Surds, it is manifeſt, that no other Product can ariſe. For the Square of the greater Member is $\frac{A+\sqrt{AA-BB}}{2}$, and of the leſſer $\frac{A-\sqrt{AA-BB}}{2}$, the Sum of which $=A$, but the double Rectangle, or $\sqrt{\frac{A+\sqrt{AA-BB}}{2}}$ in $\sqrt{\frac{A-\sqrt{AA-BB}}{2}}$ taken twice, will only give B.

For by the common Rules, if we take the Root of the Product of the parts which are comprehended under the *Univerſal* Sign, it will be equal to the Product of the *Univerſal* Roots, and therefore this twice taken will give the double Rectangle. But $\frac{A+\sqrt{AA-BB}}{2}$ in $\frac{A-\sqrt{AA-BB}}{2} =$

$$\frac{AA+A\sqrt{AA-BB}-A\sqrt{AA-BB}-\overline{AA-BB}}{4}$$

$=\frac{AA-\overline{AA-BB}}{4}=\frac{BB}{4}$, whoſe Root $=\frac{B}{2}$, and its double $=B$. Therefore the thing is plain.

But the *Inveſtigation* of this Rule may proceed after this manner. Suppoſe the Binomial $p+q$, whoſe greater Name is p. In a Square, the Squares of the two parts of the Root, will always be greater than the remaining Member. For the Terms themſelves, (ſetting aſide the Numbers) are continued Proportionals. And in three continued Proportionals, as ee, ne, nn, the Sum of the extremes $ee+nn$ is always greater than the intermediate Term $2ne$. And for that Reaſon we ought to conſider the *Name* p, as the Sum of the Squares of the two parts of the Root. Suppoſe one part of the Root $=x$, therefore $\sqrt{p-xx}=$ the other, therefore the Square of this Expreſſion $x+\sqrt{p-xx}$, that is, $p+2x\sqrt{p-xx}=p+q$, or $2x\sqrt{p-xx}=q$. Whence $4pxx-4x^4=qq$, or $4x^4-4pxx=-qq$, and ſuppoſing $xx=y$, it will be $yy-py=-\frac{qq}{4}$, and therefore $yy-py+\frac{1}{4}pp=\frac{pp-qq}{4}$, and $y-\frac{1}{2}p=\sqrt{\frac{pp-qq}{4}}=\frac{\sqrt{pp-qq}}{2}$, therefore $y=\frac{p+\sqrt{pp-qq}}{2}$. And ſince it is $y=xx=$ the Square of one part of the Root, and $p-xx=$ the Square of the other, which is therefore equal to $p-\frac{p-\sqrt{pp-qq}}{2}=\frac{p-\sqrt{pp-qq}}{2}$: hence $\frac{p+\sqrt{pp-qq}}{2}$ is the Square of the greater, and $\frac{p-\sqrt{pp-qq}}{2}$ the Square of the leſſer part of the Root, that is $\frac{A+\sqrt{AA-BB}}{2}$, and $\frac{A-\sqrt{AA-BB}}{2}$, writing A and B for *p* and *q*; as the abovementioned Rule directs us.

The ſame may alſo gathered from another Principle. Suppoſe any Square $n^2+2nm+m^2$ generated from the Root $n+m$. Take *the Sum* of the Square of the parts of the Root for the *firſt* Member, and the double Rectangle for the *ſecond*, ſo that taken ſeparately they are n^2+m^2 and $2nm$. If the Square of the *ſecond* Member be taken from the Square of the *firſt*, the Square Root of the Remainder will be the Difference of the Squares of the parts of the Root. That is, $n^4+2n^2m^2+m^4-4n^2m^2=n^4-2n^2m^2+m^4$, whoſe Root $=n^2-m^2$. But $\frac{n^2+m^2}{2}+\frac{n^2-m^2}{2}=n^2$, and $\frac{n^2+m^2}{2}-\frac{n^2+m^2}{2}=m^2$. For the Half of the *Sum* and the *Difference* of any two Quantities, if they be added to, or taken from one another, makes up thoſe Quantities. Suppoſing now

$n^2+m^2=d^2$, it will be therefore

$\frac{d^2+\sqrt{n^4-2n^2m^2+m^4}}{2}=n^2$, and

$\frac{d^2-\sqrt{n^4-2n^2m^2+m^4}}{2}=m^2$, or by ſubſtituting Equals it is $\frac{d^2+\sqrt{d^2-4n^2m^2}}{2}=n^2$, and $\frac{d^2-\sqrt{d^2-4n^2m^2}}{2}=m^2$, which Expreſſions are the ſame with the *former*.

CHAP. III.

Concerning the way of Separating and Expreſſing Unknown Quantities, when Equations are propos'd which contain divers.

IN order to this, it is neceſſary, that the young Beginner ſhould firſt of all conſider theſe two things. Firſt, *Whether the Equations propos'd are really diſtinct, and depend not on one another?* Secondly, *Whether they are more, or fewer, or equal in Numbers with the unknown Quantities, which are contained in them.* For when there are as *many* Equations as unknown Quantities, and *vice verſa*, then all of them may be reduced to one Equation, in which one unknown Quantity only ſhall remain.

That is, all the unknown may be caſt out by convenient Subſtitutions, one only excepted, whoſe value will be had in *pure known* Quantities; and then by the *given Relations* of the reſt which are unknown to this one, *they* will alſo be eaſily diſcover'd. *If there are more Equations* than unknown Quantities, then by reaſon of the ſuperfluous Number of Equations, various limitations of the Quantities among themſelves will neceſſarily ariſe, and (from ſome repugnant Suppoſitions) very frequently a Contradiction or an Impoſſibility.

But *if there are more unknown Quantities* than Equations, nevertheleſs one or more of the unknown may always be taken away, and ſo an Equation may be found by which the Relations of the others may be determin'd.

For when the Operation is finiſh'd, one or more unknown will remain, becauſe by the help of one Equation, only one unknown can be taken away. And therefore ſo many can never be taken away, as are comprehended in the Equations; ſince by the *Hypotheſis* their Number exceeds the Number of Equations propos'd. But the greater that exceſs is, the more of the unknown Quantities will remain at laſt. And the Number of them which do ſo remain, will be equal to the Sum of *Unity* and the *Difference* aforeſaid; that is if the unknown are more by one than the Equations, two will remain if by two, three, and ſo on.

But we ſhall now produce ſome Examples, by which theſe General things may be better underſtood.

And to begin with the moſt eaſie, in the firſt place we ſhall treat of Equations, in which *the unknown Quantities are neither multiply'd into one another, nor aſcend to any Power.* Which again may be divided into two different Claſſes. For either the Equations propos'd are ſo order'd, that the ſeveral unknown Quantities are found in the ſeveral Equations; or on the contrary, that ſome of them may be found in ſome Equations only, and not in others.

In *the firſt Caſe* the Operation may be generally perform'd after this manner.

Let the two Values of any unknown Quantity (which we will call a *Firſt*) equal one another, and then let there be found the expreſſion of a *Second* unknown, partly in *known* Terms, and partly in Terms *which include the other unknown.* Then repeating the Operation, find out another ſimilar Relation of this *Second* unknown, to certain Terms compounded of known and unknown Quantities. Whence making an Equation between the two found Values of this *ſecond* unknown, you will have the expreſſion of a *Third* unknown, in Terms which conſiſt of known and *fewer* unknown Quantities than before. And by the two Values of this *Third*, will in like manner be expreſt a *Fourth* unknown, in given Quantities and ſtill *fewer* unknown. And ſo on, let the number of the Equations propos'd be never ſo great.

For there will at laſt ariſe an Equation which comprehends but one unknown (with *given* Quantities) if ſo be there are as many unknown, as Equations, and ſo *vice verſa.* And this way of arguing is *General*, Certain and Perſpicuous; granting in the mean time, in ſome particular Caſes, that the matter may be perform'd in a leſſer compaſs. But it is not our buſineſs here to give *Compendiums*, nor is it ſo neceſſary for young Beginners; ſince *uſe* and *exerciſe*, in a quick and diligent Apprehenſion, will over and above ſupply all thoſe Aſſiſtances of inferior Conſideration. They ought rather to be ſhewn a certain, conſtant and uniform Rule, by the help of which, they might be able to diſcover and expreſs the unknown Quantities in all Caſes.

Suppoſe the Equations $\left.\begin{matrix} mx+ny=p \\ bx+cy=q \end{matrix}\right\}$

$$\text{From}\;\left.\begin{array}{l}\text{the Firſt}\\ \text{the Second}\end{array}\right\}\;\text{ariſes}\;\left\{\begin{array}{l}x=\dfrac{p-ny}{m}\\[2ex] x=\dfrac{q-cy}{b}\end{array}\right.$$

∴ From both $\frac{p-ny}{m}=\frac{q-cy}{b}$, or by reducing $y=\frac{bp-mq}{n-c}$. Whence x is known by the help of either Equation, which expreſſes its value in y and known Terms.

We are to obſerve in theſe General Examples, that the Coefficients and other Quantities (which are look'd upon as known, and are of an undetermin'd magnitude) *may be tranſpos'd at pleaſure*, which is not done in Examples *of Numbers*. For the chief deſign we have in view, is that the Unknown Quantities may be truly and aptly expreſt in the terms of the Equations propos'd.

$$\text{Suppoſe Eq.}\quad\left.\begin{array}{l}dx=yz\\ x+y=\dfrac{z}{n}\\ px=y+z\end{array}\right\}$$

$$\text{From}\;\left.\begin{array}{l}\text{the Firſt}\\ \text{the Second}\end{array}\right\}\;\text{ariſes}\;\left\{\begin{array}{l}x=\dfrac{yz}{d}\\[2ex] x=\dfrac{z}{n}-y\end{array}\right.$$

∴ From both $\frac{yz}{d} = \frac{z}{n} - y$, or $dz - dny = nyz$, that is $y = \frac{dz}{nz + dn}$, and therefore y is obtain'd by z and *known* Quantities.

$$\text{From } \left\{ \begin{array}{l} \text{the Second} \\ \text{the Third} \end{array} \right\} \text{ is } \left\{ \begin{array}{l} x = \frac{z}{n} - y \text{ (as before)} \\ x = \frac{y + z}{p} \end{array} \right.$$

Whence jointly $\frac{z}{n} - y = \frac{y + z}{p}$ or by reducing $y = \frac{pz - nz}{pn + n}$, which is again expreſt by z and known Quantities. Therefore $\frac{dz}{nz + dn} = \frac{pz - nz}{pn + n}$ or $pndz + ndz = pnzz - nnzz + pndz - dnzz$, that is caſting away $pndz$ on both ſides, and dividing the reſt by nz, you will have the expreſſion of the Quantity z free from all unknown Quantities.

$$\text{Suppoſe Eq. } \left. \begin{array}{l} nx + my - pz = a \\ rx - qy + sz = b \\ gx + hy - lz = c \end{array} \right\}$$

$$\text{From } \begin{array}{l} \text{the First} \\ \text{the Second} \end{array} \left\{ \text{we have} \right\} \left\{ \begin{array}{l} x = \frac{a + pz - my}{n} \\ x = \frac{b + qy - sz}{r} \end{array} \right\}$$

∴ From the firſt and ſecond *jointly*, we have z, by y and known Quantities, that is $ra+rpz-rmy=nb+nqy-nsz$, or $z=\frac{nb-ra+\overline{nq+rm}\times y}{rp+ns}$. Now another ſimiliar expreſſion of the Quantity z is to be inquir'd. Therefore from $\left.\begin{matrix}\text{the firſt}\\ \text{the ſecond}\end{matrix}\right\}$ (for they are taken at liberty, ſince we might as well have taken the 2d and 3d) ariſes

$$\left\{\begin{matrix} x=\frac{a+pz-my}{n}\text{(as before)}\\ x=\frac{c+lz-hy}{g}\end{matrix}\right\}\ \therefore\ \text{From the 1ſt and 3d, joyntly,}$$

we have another value of z, by y, and known Quantities, *viz.* $ga+gpz-gmy=nc+lzn-hny$, or $z=\frac{ga-nc+\overline{hn-gm}\times y}{ln-gp}$. Which values of the Quantity z, being equal'd to one another, y is had in pure known Quantities. And by returning to either Equation, by which z is expreſt by y and known Quantities, z is alſo known, and by the Equations firſt given, x.

$$\text{Suppoſe Eq.}\quad \left.\begin{matrix} x+nz+ny+nv=b\\ z+mx+my+mv=c\\ y+px+pz+pv=d\\ v+qx+qz+qy=f\end{matrix}\right\}$$

$$\text{From the}\ \begin{matrix}\text{1ſt,}\\ \text{2d,}\end{matrix}\left\{\begin{matrix} x=b-nz-ny-nv\\ x=\frac{c-my-mv-z}{m}\end{matrix}\right\}$$

∵ From the firſt and ſecond jointly, we have z, expreſt by y, v, and known Quantities, that is

$$z = \frac{mb - c + \overline{m - mn} \times y + \overline{m - mn} \times v}{mn - 1}$$

And that we may have another way of expreſſing z by y or v, and known Quantities,

$$\text{It will be from } \begin{matrix} \text{2d} \\ \text{3d} \end{matrix} \left\{ \begin{array}{l} x = \dfrac{c - my - mv - z}{m} \text{ (as before)} \\[2ex] x = \dfrac{d - pz - pv - y}{p} \end{array} \right\}$$

∴ From the ſecond and third jointly, ariſes the *ſecond* value of the quantity z, in y alone, and known Quantities, that is $z = \dfrac{md - pc + \overline{pm - m} \times y}{mp - p}$ (for the term mpv, is thrown away by reaſon of the contrary Signs) yet notwithſtanding we might have had the expreſſion of z, in y, v, and known Quantities, tho' that term had been there: Therefore from theſe two ways of expreſſing the Quantity z, ariſes an Equation, by which y is *firſt* expreſt, by v and known Terms, *viz.* this

$$\frac{mb - c + \overline{m - mn} \times y + \overline{m - m} \times v}{mn - 1} = \frac{md - pc + \overline{pm - mn} \times v}{mp - p}$$

After the ſame manner equalling the two values of the Quantity *x*, being taken from the third and fourth Equations, will be had a *third* value of *z*, in *v* and known Quantities; *viz.* $z = \frac{mf - qc + \overline{mq - m} \times v}{mq - q}$. And by the two other values of the Quantity *x*, by the help of the firſt and fourth Equations, *Fourthly* is expreſt *z* in *y v*, and known Quantities, that is,

$$z = \frac{bq - f + \overline{q - nq} \times y + \overline{1 - nq} \times v.}{qn - q}$$

Whence by making the two latter expreſſions of the Quantity *z* equal one to another, it will be $\frac{mf - qc + \overline{mq - m} \times v}{mq - q} =$
$\frac{bq - f + \overline{q - nq} \times y + \overline{1 - nq} \times v}{qn - q}$: by which *ſecondly* is expreſt *y* in *v* and known Quantities. So that now are found two different ways of expreſſing the unknown Quantity *y*, in the Terms of *v* and given Quantities. Whence is found the unknown Quantity *v*; and by the help of the former Equations, the others are alſo known.

When it happens that the ſeveral unknown Quantities are not found in the ſeveral Equations, we may yet diſcover the expreſſions of them by a way not much different: That is, by finding divers ways by which the ſame Unknown may be expreſt in the Terms of the other Unknown and given Quantities.

As if for Example, $$\left.\begin{array}{r} px-y=a \\ ny-mz=b \\ z+nv=c \\ v-qx=d \end{array}\right\}$$

I take from the firſt and fourth Equation the two Values of the Quantity x, *viz.* $x=\frac{a+y}{p}=\frac{v-d}{q}$ whence $y=\frac{pv-pd-qa}{q}$. Secondly, from the ſecond and fourth is had a double expreſſion of the Quantity z, *viz.* $=\frac{ny-b}{m}=c-nv$. Whence $y=\frac{mc-mnv+b}{n}$; ſo that y is now *twice* expreſt by v and known Quantities. And therefore the expreſſion of the Quantity v is found, whence the reſt are known by the help of the given Equations.

Again, ſuppoſe Eq. $$\begin{array}{ll} x+y+z+v=b & \\ x+d & =y-d \\ x+d & =dz \\ x+d & =\frac{v}{d} \end{array}$$

From the $\left\{\begin{array}{l}\text{Firſt}\\ \text{Second}\end{array}\right\}$ we have $\left.\begin{array}{l} x=b-y-z-v \\ x=y-2d \end{array}\right\}$

∴ From both ariſes $y=\frac{b+2d-z-v}{2}$.

From the $\left.\begin{matrix}\text{third}\\ \text{fourth}\end{matrix}\right\{$ it will be $dz - d\ (=x) = \frac{v}{d} - d$, and therefore $z = \frac{v}{dd}$, whence throwing out z, it will be in the former Equation $y = \dfrac{b + 2d - \frac{v}{dd} - v}{2}$ Let there be enquir'd another ſimilar value of the Quantity y.

For Example from the $\left.\begin{matrix}\text{ſecond}\\ \text{fourth}\end{matrix}\right\}$ is $\Big\}\ y - 2d$ $(=x) = \frac{v}{d} - d$, or $y = \frac{v}{d} + d$. And equalling thoſe Values of the Quantities y which are found, it will be $\dfrac{b + 2d - \frac{v}{dd} - v}{2} = \frac{v}{d} + d$. Whence v is given, and conſequently the reſt.

If the given Equations include *any Powers* of the Unknown Quantities, to diſcover the expreſſions of them, we may frequently make uſe of the *Extraction of the Roots of adfected Equations*, but ſometimes without it. For Example,

$$\text{Suppoſe Eq.}\quad \left.\begin{matrix} x + y + \frac{yy}{x} = a \\ xx + yy + \frac{y^4}{xx} = b \end{matrix}\right\}$$

$$\begin{matrix}\text{It will be firſt}\\ \text{from the ſecond}\end{matrix}\left\{\begin{matrix} xx + yy = ax - xy \\ x^4 + xxyy + y^4 = bxx \end{matrix}\right\}\begin{matrix}\text{or by adding}\\ \text{on both ſides}\\ xxyy\end{matrix}$$

$$\left\{\begin{matrix} x^4 + 2xxyy + y^4 = bxx + xxyy, \\ \therefore\ \text{extr. rad. quad.}\ x^2 + y^2 = \sqrt{bxx + xxyy} \end{matrix}\right.$$

whence, ſince we have a double expreſſion of the

Quantity $xx+yy$, then will $ax-xy=\sqrt{bxx+xxyy}$ or $aaxx-2axxy+xxyy=bxx+xxyy$, $\therefore aaxx-2axxy=bxx$, or $\frac{aa-b}{2a}=y$. Whence is eaſily had the expreſſion of x, by the Solution of an *adfected* Equation, $yy=ax-xy-xx$, in which y is known.

Suppoſe Eq. $\left.\begin{array}{l} xx-yy=n \\ x^4-y^4=m \end{array}\right\}$

Becauſe there is no Power nor Product, the two different values of which may conſtitute an Equation: Therefore make the Square of the firſt expreſſion $xx-yy$, *viz.* $x^4-2xxyy+y^4$, for thus x will be found of the ſame dimenſions on both ſides; ſo that now we have theſe two Equations,

$$\left.\begin{array}{ll} x^4-2xxyy+y^4 & =n^2 \\ x^4-y^4 & =m \end{array}\right\}$$

Whence from the $\begin{array}{l} \text{Firſt,} \\ \text{Second} \end{array}\left\{\begin{array}{l} y^4=n^2-x^4+2xxyy \\ y_4=x^4-m \end{array}\right\}$

$\therefore$ From both $\frac{n^2+m}{2}=x_4-xxyy=\overline{xx-yy}$ *

$xx=nxx$, that is, $xx=\frac{n^2+m}{2n}$. And by the help of the firſt Equation is y expreſt, becauſe $xx-n=yy$.

Again let the Eq. $\left.\begin{array}{l} xx+yy=b \\ xy+x+y=c \end{array}\right\}\begin{array}{l} \therefore xx+2xy+yy \\ =b+2xy \\ \therefore xy=c-x-y \end{array}$

Suppoſe $x+y=z$, it will be $x=z-y$, & $xy=zy-yy$. Then by ſubſtitution,

It will be inſtead of the 1ſt, 2d, $\left\{\begin{array}{l} zz=b+2zy-2yy \\ zy-yy=c-z \end{array}\right\}$

And by making equal the two values of the Quantity $zy-yy$, we have the Equation $\frac{zz-b}{2}=c-z$, or $zz+2z=2c+b$. By the reſolution of which Equation, z is expreſt in pure known Quantities. Whence by reſolving the Equation $zy-yy=c-z$ (in which z is a known Quantity) y alſo is known. And becauſe $z=y+x$, therefore will x alſo be expreſt in given Quantities.

But ſometimes the Expreſſions of the unknown Quantities ariſe to much higher Equations.

As if $\left.\begin{array}{l} bbx-aby=aab+axy \\ \&\ bxc+ayy=2aac \end{array}\right\}$

It will be from the 1ſt, 2d, $\left\{\begin{array}{l} x=\frac{aby+aab}{bb-ay} \\ x=\frac{2aac-ayy}{bc} \end{array}\right.$

Which values of the Quantity x, being equal one to another, and the Terms being rightly order'd and diſpos'd, an Equation ariſes for finding out the Quantity y, *viz.*

$$y^3-\frac{bb}{a}yy-\frac{\overline{2aac-bbc}}{a}\times y+bbc=0.$$

After the ſame manner if $\left.\begin{array}{l} evv+eev=x^3 \\ \text{and } vx-ev=ex \end{array}\right\}$

It will be $v=\frac{ex}{x-e}$, from the ſecond Equ. and ſubſtituting this Expreſſion inſtead of v in the firſt, it will be $\frac{e^3xx}{x^2-2xe+ee}+\frac{e^3x}{x-e}=x^3$; whence multiplying by $x^2-2xe+ee$, we have $2e^3xx-e^4x=x^5-2x^4e+x^3ee$ or ordering and reducing, $x^4-2x^3e+xxee-2xe^3+e^4=0$.

$$\left.\begin{array}{ll}\text{Alſo let} & zvv=c^3\\ \text{and} & zz+vv+az=dv\end{array}\right\}$$

it will be $vv=\frac{c^3}{z}$ and therefore $zz+\frac{c^3}{z}+az=d\sqrt{\frac{c^3}{z}}$ that is $z^3+c^3+azz=d\sqrt{c^3z}$, whence will ariſe an Equation of ſix Dimenſions, by which the Term z will be expreſt. But this will be done more compendiouſly, if the value not of vv, but of z, be taken from the firſt, and written down in the ſecond Equation. For ſince $z=\frac{c^3}{vv}$, therefore $\frac{c^6}{v^4}+vv+\frac{ac^3}{vv}=dv$, and reducing $c^6+v^6+ac^3vv=dv^5$, or $v^6-dv^5+ac^3vv+c^6=0$.

So that from this Example only it appears how much *an apt and convenient ſubſtitution conduces to the ſimplicity of the Expreſſion.*

And from theſe Examples, we may proceed to others, where it is not to be done without much more complicated Equations, and thoſe running out to infinite Series.

As if $\left.\begin{array}{l} \frac{x^n}{m} - \frac{y^m}{n} = a \\ my^q + z^p = b \\ \frac{z^m}{n} - v^r = c \\ x^r - \frac{v^m}{p} = d \end{array}\right\}$

For extracting on both ſides the Roots *n* & *r*,

It will be from the 1ſt, 4th, $\left\{\begin{array}{l} \left.\overline{ma + \frac{my^m}{n}}\right|^{\frac{1}{n}} = x \\ \left.\overline{d + \frac{1}{p}v}\right|^{\frac{1}{r}} = x \end{array}\right.$

And raiſing theſe Equals to the Power *r*, (for Example) it will be

$\left.\overline{ma + \frac{my^m}{n}}\right|^{\frac{r}{n}} - d = \frac{1}{p}\,v^m$, or $v =$

$\left.\overline{\frac{\left.\overline{ma + \frac{my_m}{n}}\right|^{\frac{r}{n}} - d}{\frac{1}{p}}}\right|^{\frac{1}{m}}$; which is the *firſt* Expreſſion

of the unknown Quantity *v*, in the terms of *y*, and given Quantities.

Again from the 2d, 3d, $\left\{\begin{array}{l} \left.\overline{b - my_q}\right|^{\frac{1}{p}} = z \\ \left.\overline{nc + nv^r}\right|^{\frac{1}{m}} = z \end{array}\right.$

And working as before, you will find a *ſecond* Expreſſion of v in y, and known Quantities.

$$\text{viz. } v = \overline{\frac{\overline{b - my^p}^{\frac{m}{p}} - nc}{n}}^{\frac{1}{r}}.$$

Whence, a *Finite* or *Infinite Series* (for the Exponents may be taken divers ways) *conſiſting of the Powers* of the unknown Quantity y, and given Quantities, will be equal to another *Finite* or *Infinite Series*, which likewiſe conſiſts *of the Powers of y and given Quantities*. And conſequently there will ariſe an Equation by which the unknown Quantity y, is fully determin'd and expreſt. And from hence it will be eaſy to find out Equations for determining the values of the other unknown Quantities.

This may be done when the unknown Quantities are ſo diſpos'd, *that ſome of them are firſt to be expreſt by infinite Series in the Terms of the others*, before an Equation will come forth for one unknown. For ſuppoſe the Equations,

$$\left.\begin{array}{l} 1+z = \overline{1+y}^{n} \\ 1+y = \overline{1+x}^{m} \\ 1+x = \overline{1+v}^{r} \\ 1+v = \overline{1+z}^{s} \end{array}\right\}$$

Then by comparing the two values of z, taken from the firſt and fourth Quantities.

From 1ſt, the 4th, $\left\{\text{it will be}\right.$ $z = \overline{1+y}^{n} - 1$, $z = \overline{1+v}^{\frac{1}{s}} - 1$ $\left\{\right.$ and therefore $\overline{1+y}^{n} = \overline{1+v}^{\frac{1}{s}}$

Or by expreſſing theſe Powers by their proper *Series*, it will be $1+ny+n\times\frac{\overline{n-1}}{2}y^2+n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}y^3$, *&c.* $=1+\frac{1}{s}v+\frac{1}{s}\times\frac{\overline{1-s}}{2s}vv+\frac{1}{s}\times\frac{\overline{1-s}}{2s}\times\frac{\overline{1-2s}}{3s}v^3$, *&c.* And conſequently *there will be had a Series compounded of the Powers of y, equal to a Series compounded of the Powers of v.* But by the Method of extracting a Root from an Infinite Equation (which that *excellent Mathematician* Mr. *de Moivre* has communicated to the Publick) the Quantity *y* may be expreſt by the Powers of *v*. For 'twill be

$$y=\frac{1}{ns}v+\frac{\frac{1}{s}\times\frac{\overline{1-s}}{2s}-n\times\frac{\overline{n-1}}{2}AA}{n}vv+\frac{\frac{1}{s}\times\frac{\overline{1-s}}{2s}\times\frac{\overline{1-2s}}{3s}-\overline{n\times n-1}\,AB-n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}A^3}{n}v^3$$

&c. Where the Let. $A=\frac{1}{ns}$, $B=\frac{\frac{1}{s}\times\frac{\overline{1-s}}{2s}-n\times\frac{\overline{n-1}}{2}AA}{n}$ and ſo on; Putting always inſtead of the Capital Letters B, C, D, E, *&c.* the whole Co-efficients of the Terms immediately preceding. And therefore the value of *y* may be thus otherwiſe expreſt, for $y=\frac{1}{ns}v+\frac{1-ns}{2s^2n^2}vv+$ *&c.*

After the ſame manner from Eq.

$$\begin{array}{l} 2^{d}\\ \\ 3^{d}\end{array}\left\{\begin{array}{l} x=\overline{1+v}^{\,r}-1\\ x=\overline{1+y}^{\frac{1}{m}}-1\\ \text{theref. } \overline{1+v}^{\,r}=1+y^{\frac{1}{m}}.\end{array}\right.$$

Whence twill be $1+rv+r\times\frac{\overline{r-1}}{2}v^{2}+r\times\frac{\overline{r-1}}{2}\times\frac{\overline{r-2}}{3}v^{3}$, &c. $=1+\frac{1}{m}y+\frac{1}{m}\times\frac{\overline{1-m}}{2m}yy+\frac{1}{m}\times\frac{\overline{1-m}}{2m}\times\frac{\overline{1-2m}}{3m}y^{3}$, &c.

Wherefore a *Series* compounded of the Powers of y, will be be equal to a *Series* in like manner conſiſting of the Powers of the unknown quantity v. And by help of the *foreſaid Method*, y will be expreſt in the Terms of v. That is,

$$y=mr\times v+\frac{r\times\frac{\overline{r-1}}{2}-\frac{1}{m}\times\frac{\overline{1-m}}{2m}AA}{\frac{1}{m}}vv+\frac{r\times\frac{\overline{r-1}}{2}\times\frac{\overline{r-2}}{3}-\frac{1}{m}\times\frac{\overline{1-m}}{m}AB-\frac{1}{m}\times\frac{\overline{1-m}}{2m}\times\frac{\overline{1-2m}}{3m}A^{3}}{\frac{1}{m}}v^{3}$$

&c. or alſo $y=mr\times v+\frac{\overline{m^{2}r^{2}-mr}}{2}vv$, &c.

Wherefore ſince y is had twice expreſt in the Terms of v: the Equation deſir'd, by which the quantity v is determin'd, will be had out of theſe two Expreſſions; whence it is eaſie to find out z, x and y.

And after the ſame way, in other Caſes, where the unknown Quantities are intangled one with another, after very different manners, we ſhall be able to have the Expreſſions of them. But things of this Nature neither concern vulgar Practiſe, nor are they adapted to the Capacity of Young Beginners, and conſequently improper in this place; yet I thought it would not be amiſs, if when there was an Occaſion offer'd, I ſhould ſpeak ſome things tranſiently, on a Subject neither common nor uſeleſs.

And ſo much concerning the manner of finding out Equations, by help of which, the unknown Quantities are to be expreſt, *when their Number is equal to the Number of Equations.*

But if there are more unknown Quantities than Equations, then ſome of them (as we ſaid before) will remain laſt. For if there are more than *one*, *two* will remain; if than *two*, *three*, if than *three*, *four*, and ſo on. Which will be manifeſt, both from the Nature of the thing, and by Examples. For let the Equations be, $\frac{xy}{c}=z$, and $cy+zx=cc$. Then ſubſtituting for x, either y or z, in the ſecond Equation, or $\frac{cz}{y}$, or $\frac{cz}{x}$, or $\frac{xy}{c}$, from the Firſt; there will ariſe $cy+\frac{czz}{y}=cc$, or $\frac{ccz}{x}+zx=cc$, or $cy+\frac{xxy}{c}=cc$; in every one of which, the Number of the unknown Quantities which remain at laſt is 2, *viz. the Sum of Unity*, and the exceſs of the Number of the unknown Quantities, above the Number of Equations, for the firſt was 3, and the laſt 2.

In like manner if $\frac{yy}{z} - 3a = 3x$, $av + vx = 5zz$, $cc - yy = bv$. It will be firſt $yy = 3zx + 3az$. Secondly $yy = cc - bv$, $\therefore$ $3zx + 3az = cc - bv$, and $\frac{cc - bv}{3z} = a + x$; and ſubſtituting $\frac{cc - bx.}{3z}$ in the place of $a + x$ (in the Equation $av + xv = 5zz$) it will be $ccv - bvv = 15z^3$; where two unknown Quantities only remain.

But if only two of theſe Equations were to be made uſe of, *for Example*, the *Firſt* and the *Laſt*, then in the Equation $3zx + 3az - cc + bv = 0$, there would remain three unknown Quantities at laſt, *viz.* x, z and v. Or taking the *ſecond* and *laſt* Equations, it would be $\frac{5zz}{a + x}$ $(= v) = \frac{cc - yy}{b}$, or $5bzz - acc + ayy - xcc + xyy = 0$; in which alſo there are three unknown Quantities.

Laſtly, *If there are more Equations than unknown Quantities*, there will ariſe (as we hinted before) either *Impoſſible Caſes*, or ſuch as are *determin'd* and *reſtrain'd* to ſome certain proportions of the Quantities which compoſe the Equations. Which will appear by one or two Examples.

As if ſuppoſe
$$\left.\begin{array}{l} a + 3z = b \\ x - 2b = 4a \\ x + z = 2a - b \end{array}\right\}$$

It will be (by the 3d) $x=2a-b-z$, and (by the 2d) $x=4a+2b$; $\therefore 4a+2b=2a-b-z$; (but by the 1ſt) $z=\frac{b-a}{3}$, $\therefore 4a+2b=2a-b\frac{-b+a}{3}=\frac{7a-4b}{3}$, ſo that now it is $\frac{7a-4b}{3}=4a+2b$, or $7a-4b=12a+6b$, and $7a=12a+10b$. Which is abſurd and impoſſible.

Again ſuppoſe $\left.\begin{matrix} a-y=d \\ 3y-a=c \end{matrix}\right\}$

it will be $a-d=y=\frac{c+a}{3}$, whence $3a-3d=c+a$, or $2a-3d=c$. And for that reaſon, (if the *firſt* Equation being ſuppos'd, the quantity c in the *ſecond*, be reſtrain'd to this value) the two *Hypotheſes* which were made in the beginning will agree and conſiſt with one another. For becauſe by the firſt Equation it is $a-y=d$, it will alſo be $3a-3y=3d$. But by the *ſecond* it is $3y-a=c$, and ſince c (by the Hypotheſis) $=2a-3d$, it will alſo be $3y-a=2a-3d$, or $3y=3a-3d$, or changing the Signs $3a-3y=3d$; which Equation is the very ſame with the former.

And from theſe things the Young Students may receive ſome Light, by the help of which, they may advantageouſly manage Equations of this kind. But as yet I hope it will be an acceptable piece of Service to them, if I tranſcribe thoſe elegant Rules for throwing out unknown Quantities, (which the *Incomparable* Sir *Iſaac Newton* has given us) and put them down in his own Words.

Rule I. *When the Quantity to be thrown out is of one Dimenſion only in both the Equations, both its values are to be inquir'd by the Rules* (*viz.* of Reduction) *before deliver'd, and one to be equal'd to the other.*

Rule II. *When in one of the Equations at leaſt, the Quantity to be thrown out is only of one Dimenſion, the value of it is to be inquir'd in that, and to be ſubſtituted for it ſelf in the other Equation.*

Rule III. *When in neither of the Equations the Quantity to be thrown out is of one only Dimenſion, the value of the greateſt Power of it is to be Sought in each Equation; and then if thoſe Powers are not the ſame, the Equation which has the leſſer Power is to be multiply'd by the Quantity to be thrown out, or by its Square or Cube,* &c. *that it may be one of the ſame Power as in the other Equation. Then the values of thoſe Powers are to be ſuppos'd equal, and a new Equation will come forth, where the greateſt Power or Dimenſion of the Quantity to be thrown out is leſſen'd. And by repeating this Operation, that Quantity will at laſt be thrown out.*

Which may be eaſily apply'd, and deduc'd to Practiſe by the foregoing Examples.

CHAP.

CHAP. V.

Of the Nature of Equations.

THE Solutions of Problems, are contain'd in Equations, which Solutions are actually diſcover'd and brought forth by *Analyſis*, or extraction of Roots. But that the Method of handling Equations (uſeful both to their *Solution* and *Conſtruction*) may be the better underſtood, and that we may have the firmer Grounds for our whole future Practiſe, we will trace this Matter up to its Original, beginning with the formation of Equations.

Equations are generated (or at leaſt are ſuppos'd to be generated) by the multiplication of certain *Binomials*, in which a quantity, *which is look'd upon as unknown*, is joyn'd at Pleaſure with others which are known by the Signs + and —.

As if we conceive the unkown Quantity z ambiguouſly to ſignify any of theſe two Numbers, *viz.* 4 and 5, ſo that z is $z=4$, $z=5$, or $z-4=0$, and $z-5=0$. Therefore multiply theſe into one another, there is this Equation $zz-5z-4z+20=0$, or $zz-9z+20=0$, by ſumming up all thoſe Terms in which z is found.

In like manner, if we expound z by c and d, or by $-c$ and $-d$, or c and $-d$ ſo that it will make

$$\begin{matrix} z-c=0 \\ z-d=0 \end{matrix} \left\{ \text{or} \begin{matrix} z+c=0 \\ z+d=0 \end{matrix} \right\} \text{or laſtly}$$

$\left.\begin{array}{l} z-c=0 \\ z+d=0 \end{array}\right\}$ it will be in the *Firſt Hypotheſis*, $zz-zc+zd-dc=0$, in the Second $zz+zc+zd+dc=0$, in the Third $zz-zc+zd-dc=0$. That is, ſuppoſing $d+c=q$, $d-c=\pm p$, as d is, $>$ or $\rhd\, c$; there will ariſe $zz-qz+dc=0$, or $zz+qz+dc=0$, or $zz\pm qz-dc=0$.

So, if $x=a$, $x=b$, $x=-c$, $x=d$, &c. or $x-a=0$, $x-b=0$, $x+c=0$, $x-d=0$, &c. Then multiplying $x-a$ by $x-b$, there comes forth this Equation $xx\begin{array}{l}-a\\-b\end{array}x+ab=0$. And again multiplying this by another Binomial $x+c$, there will ariſe an Equation one degree higher; that is,

$$x^3\begin{array}{l}-a\\-b\\+c\end{array}xx\begin{array}{l}+ab\\-ac\\-bc\end{array}x+abc=0.$$

Farther multiplying this laſt by $x-d$, will be produc'd an Equation one degree higher ſtill, which is

$$x^4\begin{array}{l}-a\\-b\\+c\\-d\end{array}x^3\begin{array}{l}+ab\\-ac\\-bc\\+ad\\+bd\\-cd\end{array}xx\begin{array}{l}+abc\\-abd\\+bcd\\+acd\end{array}x-abcd=0.$$

And imagining new values of the unknown Quantity x, and thoſe either *Poſitive* or *Negative* at Pleaſure; you may go on infinitely in forming more complicated Equations.

From the Generation of Equations we may infer,

1. *In ſeveral Multiplications, the higheſt Power of the unknown Quantity advances one degree.*

Therefore, by the *firſt* Multiplication, the unknown Quantity aſcends to a *Square*, whence it is call'd a *Quadratic* Equation; by the *ſecond*, to a *Cube*, by the *third*, to a *Bi-quadratic*, *&c.* whence likewiſe it is nam'd a *Cubic* or *Bi-quadratic* Equation, *&c.*

2. *There may be as many different Values of the unknown Quantity* (or Roots of an Equation) *as the unknown Quantity has Dimenſions, and no more.*

This is plain from the Generation of Powers.

3. *The Sum of all the Terms together is equal to nought.*

For by *Hypotheſis* the ſeveral *generating Binomials*, that is, the ſeveral *Factors* of the Sum produced, are equal to nought; and therefore multiply'd into one another make nought.

4. *It is plain, if any of the Quantities aſſum'd at firſt, be ſubſtituted every where in place of the unknown Quantity, that the Sum of all the Terms will ſtill be equal to nought.*

As in the foreſaid Cubic Equation

$$x^3 \begin{array}{l} -a \\ -b \\ +c \end{array} xx \begin{array}{l} +ab \\ -ac \\ -bc \end{array} x + abc = 0;$$

Where there was $x=a$, or b, or $-c$. If a be put in its place.

there will ariſe $a^3 \begin{array}{l}-a\\-baa\\+c\end{array} \begin{array}{l}+ab\\-ac\\-bc\end{array} a+abc=0$, if b,

will be $b^3 \begin{array}{l}-a\\-b\,bb\\+c\end{array} \begin{array}{l}+ab\\-acb\\-bc\end{array} +abc=0$, but if $-c$,

it will be $-c^3 \begin{array}{l}-a\\-bcc\\+c\end{array} \left.\begin{array}{l}+ab\\-ac\\-bc\end{array}\right\} \underset{-}{\times} c+abc=0.$

Which is manifeſt, both *from the diſpoſition of the Terms* (*which is ſuch that they all deſtroy one another*) and alſo from this conſideration, *that ſince the whole Sum was before equal to nought by the Generation, it is impoſſible that an equal Subſtitution ſhould make any alteration.* But this will be prov'd otherwiſe after this manner.

For becauſe (by the *Geneſis*) is had

$$x^3+\overline{c-a-b}\times xx+\overline{ab-ac-bc}\times x+abc=0,$$

therefore by dividing the whole by x^3, it will be

$$1+\overline{c-a-b}\times\frac{1}{x}+\overline{ab-ac-bc}\times\frac{1}{xx}+\frac{1abc}{x^3}=0;$$

and multiplying by the ſame Power (which in this Example is a *Cube*) of any of them to which x was ſuppos'd equal. *For example*, by $-c^3$ will be produc'd

$$-c^3+\overline{c-a-b}\times\frac{-c^3}{x}+\overline{ab-ac-bc}\times\frac{-c^3}{xx}+$$

$\dfrac{abcx-c^3}{x^3}=0$. But the ſame is perform'd by this Multiplication, as if in the room of x you put $-c$ through the whole Equation. For becauſe $x=-c$ is $\dfrac{-c^3}{x}=+c^2$, and $\dfrac{-c^3}{xx}=-c$, and $\dfrac{-c^3}{x^3}=1$, whence ſubſtituting theſe in their places, the Sum of all will be, $-c^3+\overline{c-a-b}\times c^2+\overline{ab-ac-bc}\times -c+abc$ which is equal to nought, that is,

$$-c^3\begin{matrix}-a\\-b\\+c\end{matrix}cc\begin{matrix}+ab\\-bc\\-ac\end{matrix}\times -c+abc=0,$$ as before.

5. *Therefore that is the Root of any Equation, which (whether it be poſitive or negative) being ſubſtituted in the place of an unknown Quantity, makes the whole Sum equal to nought.*

For the Quantities which make this, are the very ſame as thoſe which the unknown Quantity x did at firſt ambiguouſly ſignifie; that is, they are its different *Values*, or, in another word, the *Roots* of the generated Equation. But beſides theſe, no other Quantities can do the ſame. *For Example*, in a Numerical Equation $xx-7x+12=0$, produc'd by the *Factors* $x-3$, and $x-4$, where by conſequence the unknown Quantity x is to be explain'd by the Number 3 or 4; if you put 4 in the place of x, it will then be $16-28+12=0$, or if 3, it will be $9-21+12=0$: But let 5 be put, and then it will be $25-35+12>0$, if 2 it will be $4-24+12<0$, if 1, it will be $1-7+12>0$.

And ſo may any other Examples, and any Numbers be try'd at pleaſure.

6. *Having chang'd the Signs in the even places, without changing the reſt, the Co-efficient, or known Quantities of the ſecond, third, fourth, fifth Terms, &c. give reſpectively the Sums of the Roots, of the* Rectangles *under two, of the* Solids *under three, of the* Contents *under four, &c.* But we are to take notice, that the laſt Term is always the Product, whether it be a *Rectangle* or a *Solid*, under all the Roots. For ſuppoſe $y=p$, $y=q$, $y=-r$, $y=-s$. And from theſe Factors $y-p=0$, $y-q=0$, $y+r=0$, $y+s=0$, let the Equation be form'd.

$$y^4 \begin{array}{l} -p\\ -q\\ +r\\ +s \end{array} y^3 \begin{array}{l} +pq\\ -pr\\ -qr\\ -ps\\ -qs\\ +rs \end{array} y^2 \begin{array}{l} +pqr\\ +pqs\\ -qrs\\ -prs \end{array} y + pqrs = 0.$$

The Co-efficient of the Second Term, as it is here affected with the Signs, is $r+s-p-q$, and the Signs being chang'd into the contraries, it is $p+q-r-s$, which is the Sum of the Roots under their proper Signs. For r and s were Negative quantities by the firſt ſuppoſition. But $pq+rs-pr-qr-ps-qs$, the known Quantity of the third Term, is the Sum of the Rectangles under any two Roots, with their proper Signs; for two of theſe, *viz.* pq and rs, are *affirmative originally*, becauſe they are produc'd by multiplying an Affirmative into an Affirmative, and a Negative into a Negative; and the other four are negative, by reaſon of multiplying the Affirmative into a Negative, from which they ariſe. Likewiſe changing the Signs of $pqr+pqs-qrs-prs$, it is $qrs+prs-pqr-pqs$, the Sum of the Solids under any three of the Roots, and thoſe affected with their proper Signs: for there are two of theſe Solids which *Originally have Affirmative Signs, viz.* qrs and prs, becauſe they ariſe from the Multiplication of an Affirmative into an Affirmative (*viz.* p & q into r) the other two, pqr, pqs are *Negatives*, becauſe on the contrary they are produc'd from the Multiplication of a *Negative* into an *Affirmative*, *viz.* pq into r and s. But the *laſt* Term $pqrs$ (*viz. the content* under all the Roots) has a poſitive Sign, with which it is *originally* affected, by reaſon of multiplying the Affirmative pq into the Affirmative rs.

And *in a Word*; have an Eye to the Signs, with which you ſuppoſe the Quantities originally affected, and how they are chang'd into their contraries, while you make Binomials in order to the multiplication; and theſe being well conſider'd, you cannot be ignorant of the Reaſons and Demonſtrations of all.

Seventhly, *If the Affirmative and Negative Members which conſtitute the known Quantity of any Term of the Equations be equal in themſelves, they will deſtroy one another, and conſequently that Term will be utterly loſt.*

And vice verſâ, *if any Term of the Equation be wanting, thoſe Members muſt neceſſarily be equal.*

And therefore (for Example) an Equation will then be without the *Second* Term, when the Poſitive Roots taken together are equal to the Negative alſo taken together. And ſo of the reſt.

Eighthly, *The ſum of an Equation may always be divided by ſome Binomial, compounded of the unknown Quantity, leſs the value of any of the Affirmative Roots, or more the value of any of the Negative ones.*

Nor will the propos'd Equation be divided by any other *Binomial.*

Further by means of the ſaid diviſion, the dimenſions of the Equation will be ſo much leſſen'd. For Example, dividing the *laſt Equation* of *four Dimenſions* by $y-p$, it will become a *Cubical* one, having one *Affirmative*, and two *Negative* Roots; and again, dividing this *Cubical* by $y+r$, you will have a *Quadratic* Equation, in which are contain'd only one Negative, and one Affirmative: and this very Quadratic being again divided by $y+s$, there reſults a ſimple Equation $y-q=0$. Thus alſo, the Equation $x^3-6x^2-51x+280=0$, generated by the factors $x-5=0$, $x-8=0$, $x+y=0$, may be divided by any of them without a remainder, but not by $x+$ or $-$, any other Number. For Example, dividing it by $x+7$, will ariſe $xx-13x+40=0$, an Equation containing the two remaining Roots, *viz.* the Affirmative 8 and 5; which being again divided by $x-5=0$, or $x-8=0$, will be had in the Quotient $x-8=0$, or $x-5=0$.

Hence, if the Sum of an Equation cannot be divided by a Binomial compounded of an unknown quantity $+$ or $-$ any other given quantity; that given quantity is not the value of the Roots of that Equation.

Ninthly, *It appears alſo by the Geneſis, how in Equations* (all the Roots of which are *Real*) *the*

Numbers of the Roots, both Affirmative and Negative, may be defin'd.

For there are as many *Affirmative* Roots, as, in a continued *Series*, there are tranſmutations of the Signs $+$ into $-$, and $-$ into $+$, and conſequently the reſt are *Negative:* or (which is the ſame) there are as many Negative, as there are pairs of Signs $-$ and $-$ or $+$ and $+$, immediately following one another. So in the Equation $x^3-6x^2-51x+280=0$, there are two Affirmative Roots, by reaſon of two changes of the Signs, *viz.* of $+$ into $-$, in the Terms x^3 and $6x^2$, and of $-$ into $+$, in the Terms $51x$ and 280: One alſo is Negative, as is manifeſt without regard to the Signs.

And by the ſame Rule, in the Equation $z^4-4z^3-19z^2+106z-120=0$, are three Affirmatives and only one Negative: *For there are three variations of the Signs, and but one immediate ſucceſſion of the ſame Signs.* And the reaſons of all theſe will eaſily be deduc'd by Young Students from the nature of *Algebraic* Multiplication.

Tenthly, and Laſtly, *From the ſame principle proceeds a Rule for changing the Roots of Equations, Affirmatives into Negatives, and Negatives into Affirmatives.*

That is, by changing the Signs in the alternate Places (as 2d, 4th, 6th, *&c.* or 1ſt, 3d, 5th, *&c.*) the reſt being not chang'd at all.

For Example, the Equation $z^4-4z^3-19z^2+106z-120=0$, contains three Affirmative Roots 2, 3, 4, and another Negative one -5. But the Signs being chang'd in even places, comes forth $z^4+4z^3-19z^2-106z-120=0$, in which therefore

fore by Power of the firſt Rule, three are Negative, *viz.* -2, -3, -4, (the ſame which before were Affirmative) and one Affirmative which is 5 (which before was Negative) that is, by this operation, the Equation is chang'd into that very form under which it ſhould appear, if by taking the Factors $z+2=0$, $z+3=0$, $z+4=0$, $z-5=0$. But the ſame would happen if the Signs were chang'd in the odd places, and therefore if for $z^4-4z^3-19z^2+106z-120=0$, write $-z^4-4z^3+19z^2+106z+120$. For here there are the ſame three Negative Roots, and the ſame Affirmative as before, only the Members which compoſe the factors are tranſpos'd, as inſtead of $z+2$, $z+3$, $z+4$, $z-5$, there are $2+z$, $3+z$, $4+z$, $5-z$, whence the higheſt Power will have a Negative Sign.

Schol. For the further explication of *the Nature of Roots* let this general diſtinction of them be obſerv'd, by which they are divided into *Real* (or Poſſible) and *Imaginary* or Impoſſible). Again, of the *Real* ſome are *Affirmative*, and ſome *Negative*, of which alone we have hitherto diſcours'd. As to the *Impoſſible*, we are to know that they are neither Affirmative nor Negative, and conſequently the Underſtanding can have no conception of them. But all Impoſſibility owes its riſe to a *Negative Square*, and is founded upon that.

For there is no Quantity given, which we can any ways imagine or explain, but if it be multiply'd into it ſelf, whether it be Negative or Affirmative, it will make an Affirmative Square. Therefore the Root of a Negative Square is ſomething Fictitious and Impoſſible: As are alſo the Roots of all Negative Powers which are deſign'd by even Numbers, as *reducible to Squares.*

Now hence riſes a neceſſary limitation, relating to one of the Rules at leaſt which were brought before, and without which it can by no means be receiv'd as univerſally true. That is, *When none of the Roots of an Equation are impoſſible, then we may ſafely judge of the Number of the Affirmative and Negative Roots by the Signs of the Equation.* Whereas it may happen, that there may not be as many Affirmative, as there are variations of the Signs + and −; nor ſo many Negatives, as thoſe Signs are found following one another.

For the Impoſſible Roots are neither Affirmative nor Negative, as was ſhown before; and therefore when an Equation ſhall include any of them, *the Rule is of no force*, except ſo far as you ſhall eſteem them for *ambiguous* ones. For the better underſtanding of this, we muſt know, that Equations may be generated by a continual Multiplications, not only of *Real* and Poſſible, but alſo of Impoſſible and *Fictitious* Quantities. As if you ſuppoſe $x=2+\sqrt{-1}$, and $x=2-\sqrt{-1}$ (*where the Impoſſible Quantity is the Root of Negative Unity*) then multip. $x-2-\sqrt{-1}=0$

by $x-2+\sqrt{-1}=0$

there will ariſe the Product $x^2-2x-x\sqrt{-1}-2x+4+2\sqrt{-1}+x\sqrt{-1}-2\sqrt{-1}+1=0$, or rejecting $x^2-4x+5=0$, which is therefore an *impoſſible Equation.* But this being again multiply'd into $x-2=0$, *for example*, it will produce the Equation $x^3-6x^2+13x-10=0$; whereof if you regard the Signs, three Affirmative Roots will appear at firſt view: but only one of them is *real*, and the other two (as you have ſeen already) *are imaginary and impoſſible.*

So the Equation $x^3 - 4xx + 7x - 6 = 0$ (in which from the variations of the Signs you would preſently imagine there were three Affirmative Roots (there ariſes from the Roots 2, $1 + \sqrt{-2}$ $1 - \sqrt{-2}$, the two latter of which have the ſame Mark of Impoſſibility.

And hence it may be further concluded, that it is not univerſally true, *That there are as many Roots of any Equation, as there are Dimenſions of the higheſt Power*; unleſs you extend the Notion of a Root *to every thing* (poſſible or impoſſible) *which you conceive does any way concur to form an Equation.* And indeed it is true, that the impoſſible Roots themſelves (as well as the reſt) being ſubſtituted every where in the place of an unknown Quantity, leave the whole Sum equal to nought, as will be plain to any one who makes tryal of it. But if you mean by a Root only this, *What may be explain'd or underſtood*, then the Rule muſt undergo a limitation, ſince otherwiſe it may plainly appear, in abundance of Caſes, that it is falſe and fallacious.

But there are one or two things to be hinted concerning theſe impoſſible Roots; in which we follow *that Great Man* Sir *Iſaac Newton*, in his Treatiſe before-mention'd.

1. *Since the Caſes of Problems, are ſometimes poſſible, but in other circumſtances altogether impoſſible; for this reaſon, the Roots of Equations ought ſometimes to be impoſſible; for otherwiſe they would repreſent all Problems as poſſible in every Caſe without diſtinction.*

So by conſtruing the Equation $zz - az + bb = 0$, where a Circle is deſcrib'd with the Radius, $= \frac{1}{2}a$, and the length of b is ſet off in the Tangent, at the end of which a perpendicular Line muſt be erected. This being produc'd, will *cut* the Circle in two Points, as long as the length of the Tangent is leſs *than the Radius: But it will touch when they are equal, and will never meet, when the latter exceeds the former.* Thus the poſſibility or impoſſibility of a Conſtruction is truly and juſtly ſhewn by the Equation, in which the Value of the unknown Quantity z is expreſt (that is, $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$) For according as we ſuppoſe $\frac{1}{2}a >$ or $< b$, ſo may the Square $\frac{1}{4}aa - bb$ be *Affirmative*, or *Negative*, and conſequently the Root of it (and ſo the whole z) is either poſſible or impoſſible.

2. *The Roots of Equations in themſelves are oftentimes Real and Poſſible, ſince they are not to be explain'd by Figures*; that is, when it is impoſſible to exhibit them by Schemes.

The reaſon of which is, that thoſe Relations which we may imagine Quantities in an Equation, to bear one to another, are too abſtracted and general, to be confin'd under the narrow limits of Figures.

As if this Equation be propos'd $\frac{y^2}{a} = x$. In theſe abſtracted Terms, there is nothing meant in this Equation, but that x is a third Proportional to the two Quantities y and a. And this will be always poſſible, the Quantity a remaining *unvary'd*, and *y varying*, nay, increaſing infinitely. But if you would explain this very Equation by the Relation of ſome Lines in a Geometrical Figure; *for Example*, a *Circle*, ſuppoſing the *Diameter a*, the *Chord* drawn from its extremity, *y*, then will *x* be the Abſciſs belonging to that *Chord*.

And here, in determining the bigneſs of the Quantity *y*, you have certain limits ſet you, by the nature of the Circle, which you cannot go beyond; for it will neceſſarily be leſs than *a*, otherwiſe you will never have, by a Circle, any third Proportional, or Abſciſs *x*.

3. But we muſt obſerve, *That as this Impoſſibility owes its riſe to the Nature and Property of a certain Figure; ſo by applying a Figure of a different Nature, it may come to paſs, that there be no Impoſſibility at all found, but that* the Geometrical deſignation *may go always even* with the Algebraic expreſſion.

So in the former Example, if for *y* you put the *ordinate* for *a* the *Parameter* of a *Parabola*; then will *x* be the Abſciſs of the Diameter, agreeing to that ordinate. And by the Nature of this Figure, whatever length you take for the Quantity *y*, you will eternally aſſign a third Proportional *x* within the Bounds of the Scheme.

4. Note, *That the Number of Impoſſible Roots is always even.*

For the Number of unequal Roots which together become *equal*, is always even; and therefore

fore the Number of them, which come forth *Impoſſible* after equality, muſt be even. But after what manner two unequal Roots, together become equal, that is, two interſections of any Curve, are at the ſame time converted into the point of Contact, the Young Student muſt be taught from *Geometry*.

Fifthly, And for this reaſon, *the Equations of uneven Dimenſions, cannot have all their Roots Impoſſible.*

Having explain'd theſe things, in treating of Equations, we now proceed to Problems or neceſſary Practiſes.

Prob. 1. *To find out how many Impoſſible Roots there are in any Equation propos'd.*

The different parts of this Rule, as they follow in order, we ſhall treat of in the Practiſe it ſelf; for ſo the whole will be more eaſily apprehended, and more deeply rooted in the Mind.

Suppoſe the Equation $x^3-4xx+4x-6=0$, in which *no Term is wanting*, and according to which, as a Standard, all the reſt (of whatever dimenſions they be) *which want no Term or only one*, are to be treated.

Write down the natural Numbers up to the Number of the dimenſions of the Equation propos'd, which here is 3, *viz.* 1. 2. 3. And over theſe write the ſame Numbers, but in an inverted Order, by which means come forth the following Fractions $\frac{3}{1}$. $\frac{2}{2}$. $\frac{1}{3}$. divide the laſt of each by the former, *viz.* $\frac{3}{1}$) $\frac{2}{2}$ ($\frac{1}{3}$, and $\frac{2}{2}$) $\frac{1}{3}$ ($\frac{1}{3}$ place the Quotients ariſing $\frac{1}{3}$

and $\frac{1}{3}$ over the middle Terms of the Equation, after this manner, $x^3 - \overset{\frac{1}{3}}{4xx} + \overset{\frac{1}{3}}{4x} - 6 = 0$. Take the Squares of the middle Terms, multiply'd into the Fractions, which are over their Heads, *viz.* $\frac{16x^4}{3}$, & $\frac{16xx}{3}$. Compare them with the Rectangles of the Terms which ſtand on both ſides, *viz.* $\frac{16x^4}{3}$ with $x^3 \times 4x$, & $\frac{16xx}{3}$ with $4xx \times 6$. According as theſe Rectangles are *greater*, or *not greater* than the Quantities aforeſaid, with which they are compar'd, ſo under the *middle correſponding Terms*, write — or +: as becauſe $4x^4 < \frac{16x^4}{3}$, & $24xx > \frac{16xx}{3}$, therefore under the Term $4xx$ I write +, and — under $4x$. And then the Equation being diſtinguiſh'd by its Signs, appears thus, *viz.* $x^3 - \underset{+}{\overset{\frac{1}{3}}{4xx}} + \underset{-}{\overset{\frac{1}{3}}{4x}} - 6 = 0$. In all caſes place the Sign + under the firſt and laſt Term of the Equation, ſo that now the whole order of the Signs under written ſtands thus, *viz.*

$$\underset{+}{x^3} - \underset{+}{\overset{\frac{1}{3}}{4xx}} + \underset{-}{\overset{\frac{1}{3}}{4x}} - \underset{+}{6} = 0.$$

Having done this, we muſt remember this Rule, there are as many Impoſſible Roots, as there are variations of theſe Signs (which are *plac'd under*) of + into — and — into +. And for that reaſon there are two Roots of this ſort in this Equation, becauſe there happen to be two variations of Signs.

So in the Equation $x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 = 0$, two Impoſſible Roots will be diſcover'd.

For it is $\frac{32}{5}x^8 > 4x^8$, & $8x^6 = 8x^6$, & $2x^4 > -20x^4$, & $10xx > 8xx$. And therefore the Signs are written under in this Order, *viz.* $+ + - + + +$.

Where any Term (but none but one) *is wanting we are to conceive a Cypher in its place)* and in the reſt proceed as above, after the ſame manner as if that Term were not wanting.

For Example, Suppoſe an Equation $x^4 * - 6xx - 3x - 2 = 0$, therefore the ſeries of the natural Numbers 1. 2. 3. 4.

And the Fractions requir'd $\frac{4}{1}$. $\frac{3}{2}$. $\frac{2}{3}$. $\frac{1}{4}$.
And the Quotients ariſing $\frac{4}{1})\frac{3}{2}(\frac{3}{8}.\frac{3}{2})\frac{2}{3}(\frac{4}{9}.\frac{2}{3})\frac{1}{4}(\frac{3}{8}$.

The Equation with the Fractions written above.

$$\begin{matrix} & \frac{3}{8} & \frac{4}{9} & \frac{3}{8} & \\ x^4 & * & -6xx & -3x & -2=0 \end{matrix}$$

The Squares of the middle Terms multiply'd into the Fractions.

$$0\times\tfrac{3}{8}=0$$
$$\frac{144x^4}{9}$$
$$\frac{27xx}{8}$$

The Rectangles of the Terms ſtanding on both ſides about the middle Terms.

$$-6x^6$$
$$0\times 3x = 0$$
$$12xx$$

The compariſon of the Rectangles with the Squares multiply'd into the Fractions.

is $-6x^6 > 0$
& $0\times 3x < \frac{144x^4}{9}$
& $12xx > \frac{27xx}{8}$

So that under * we are to place +, under $6xx$ alſo +, and under $3x$ the Sign —.

Whence ſuppoſing alſo + under the firſt and laſt Term, the Order of the Signs written under proceeds in this manner,

$$\underset{+}{x^4}\ \underset{+}{\overset{\frac{3}{8}}{*}} - \underset{+}{\overset{\frac{4}{9}}{5xx}} - \underset{-}{\overset{\frac{3}{8}}{3x}} - \underset{+}{2} = 0,$$

by which it is concluded that the Equation propos'd contains two Impoſſible Roots.

When *two or more Terms are wanting at the ſame time*, the Practiſe is to be in a method in ſome few things differing from the former. That is, under the *Firſt* of the deficient Terms write —, under the *Second* +, under the *Third* —, and ſo on, changing the Signs alternately: Unleſs *the Signs of the Terms (which ſtand next about the deficient ones) are contrary*, for then the Sign + is always to be plac'd under the laſt of thoſe which are wanting.

As in the Equation $x^5 + ax^4 + a^5 = 0$, or $x^5 + ax_4 * * * + a^5 = 0$, under x^5 and a^5 *the firſt and laſt Term of the Equation*, + is to be written, under $ax4$ likewiſe +, becauſe $\frac{2}{5} aax^8 > x^5 \times 0$, and (according to the *Tenor* of the Rule) — is to be plac'd under the firſt Term of the deficients, under the ſecond +, under the third —, becauſe the Signs of the Terms ax^4 and a^5 are not contrary. So that now having made the deſign'd diſpoſition of the Signs, the Equation ſtands thus,

$$\underset{+}{x^5} + \underset{+}{ax^4}\ \underset{-}{*}\ \underset{+}{*}\ \underset{+}{*} - + a^5 = 0:$$

in which there are four variations of the Signs, and conſequently four Impoſſible Roots. But if the Signs of the Terms $ax4$ and a^5 had been contrary, as in this

Equation $\underset{+}{x^5}\underset{+}{+ax^4}\;\underset{-}{*}\;\underset{+}{*}\;\underset{+}{*}+a^5=0$, then the laſt of the deficient Terms for that reaſon would have had the Sign $+$ written under it, and then there had been only two impoſſible Roots. And this happens alſo in the Equation following, which for the ſake of young Beginners we ſhall treat of more fully; where notwithſtanding for other reaſons it happens that ſix impoſſible Roots are found. Suppoſe for inſtance $x^7-2x^6+3x^5-2x^4+x^3 * * -3=0$.

The Series of Numbers for the Denominators. Of Fractions	1.	2.	3.	4.	5.	6.	7.
The Fractions themſelves	$\frac{7}{1}$.	$\frac{6}{2}$.	$\frac{5}{3}$.	$\frac{4}{4}$.	$\frac{3}{5}$.	$\frac{2}{6}$.	$\frac{1}{7}$.
The Quotients ariſing		$\frac{3}{7}$.	$\frac{5}{9}$.	$\frac{3}{5}$.	$\frac{3}{5}$.	$\frac{5}{9}$.	$\frac{3}{7}$.

Æquations with the Fractions aſcrib'd

$$x^7-\overset{\frac{3}{7}.}{2x^6}+\overset{\frac{5}{9}.}{3x^5}-\overset{\frac{3}{5}.}{2x^4}$$

$$\overset{\frac{3}{5}.}{+x^3}\;\overset{\frac{5}{9}.}{*}\;\overset{\frac{3}{7}.}{*}-3=0$$

The Squares of the middle Terms multiply'd into the Fractions. $\left.\right\}$ $\frac{12}{7}x^{12}.\ \frac{45}{9}x^{10}.\ \frac{12}{5}x^8.\ \frac{3}{5}x^6$

The Rectangles of thoſe Terms which are neareſt the mean Terms. $\left.\right\}$ $3x^{12}.\ 4x^{10}.\ 3x^8.\ 0.$

The Compariſon of the Squares multiply'd into the Fractions with the Rectangles
$$\left\{\begin{array}{l}\frac{12}{7} < 3.\frac{45}{9} > 4.\\[1ex] \frac{12}{5} < 3.\frac{3}{5} > 0.\end{array}\right.$$

Therefore under the ſecond Term of the *Equation* we ſhould write the Sign —, under the third +, under the fourth —, under the fifth +, under the firſt *deficient* Term —, under the ſecond (which is alſo the laſt of them) we ought to write +, for the Signs of the Terms x^2 and 3 are contrary; and laſtly under the firſt and laſt Term of the Equation, the Sign +, as has been very often hinted at. Wherefore the diſpoſition of the Signs is thus, *viz.*

$$\begin{array}{l} x^7 - 2x^6 + 3x^5 - 2x^4 + x^3 * * - 3 = 0\\ \;+ \quad - \quad + \quad - \quad + - + + \end{array}$$

by which it is ſhewn, that there is but one *real*, and ſix *impoſſible* Roots of this Equation.

Schol. 1. But we muſt take notice, that *there may be more Impoſſible Roots than this Rule will diſcover*; but this *very rarely happens*, according to the Judgment of the *Celebrated* Sir *Iſaac Newton*, to whom we owe theſe Rules.

Schol. 2. Tho' *there may be more* Impoſſible Roots than are diſcover'd by the help of this Rule; yet it ſuppoſes, that *there are as many Roots of that ſort, as there are variations of the Signs under-written*:

ten: And therefore, by this Mark (wherever it appears) it ſhews ſome Impoſſible Roots to be. *Therefore in Equations, all the Roots of which are* poſſible *or* real, *there can be no variation of the under-written Signs.* It may indeed ſo fall out, *that no variation can happen,* where notwithſtanding *there may be an Impoſſible Root* (becauſe there may be more than what the Rule diſcovers, and the Rule only judges by the variations of the Signs under-written) but where a variation happens *that an impoſſible Root ſhould not be found,* it is wholly foreign to the Senſe and Meaning of the Rule.

And therefore in Equations, which involve none but real Roots, *we are not to expect any change of the Signs under-written.*

But there will neceſſarily ariſe a variation of the Signs, if the Sign $+$ being plac'd under the *firſt* and *laſt* Term, the Signs of all the intermediate Terms be not in like manner *Affirmative.* And according to the Rule, the Signs of the middle Terms will by no means be Affirmative, *except the Squares of them, multiply'd into the Fractions over-head, perpetually exceed the Rectangles under the Terms* next on both ſides. Wherefore in Equations, *all the Roots of which are real,* the *Squares* of the middle Terms multiply'd into the Fractions over-head, *ought perpetually to exceed the Rectangles* under the Terms next on both ſides. Therefore this being univerſally demonſtrated, *from the Nature of the Co-efficients, with which the middle Terms of an Equation are adfected;* the reaſon of this Rule will be plain, which teaches us to diſcover the impoſſible Roots from a contrary Cauſe (*viz.* the deficiency of the Squares multiply'd into the Fractions from the Rectangles). But it is ſufficient (to *illuſtrate* this) to add an Example in Numbers.

Suppoſe the Equation $z^3 - 8z^2 + 17z - 10 = 0$, whoſe Roots are 5. 2. 1. all *real* and *affirmative.* Now it will be $64 \times \frac{1}{3} > 17$ and $289 \times \frac{1}{3} > 80$: ſo that under the *ſecond* and *third* Term the Sign + ſhould be written. If the Roots were 1. 2. −5, whence the Equation $x^3 + 2x^2 - 13x + 10 = 0$; in this Caſe, it is $\frac{4}{3} > -13$, and $\frac{169}{3} > 20$, and therefore the Sign + muſt again be plac'd under the the ſecond and third Term. Which would be the ſame, if the Roots were −1. −2. +5, and the Equation $x^3 - 2x^2 - 13x - 10 = 0$; or −1 −2. −5, and the Equation $x^3 + 8x^2 + 17x + 10 = 0$, as is plain upon ſight.

So taking any other Equation of whatever Dimenſions (whoſe Roots are all real) you may find the ſame.

Probl. 2. *To find whether Impoſſible Roots lie among the Affirmative or Negative ones.*

Obſerve the Signs of the Terms in the Equation, which are over the *varying Signs written under.* For there are as many *Affirmative* Roots Impoſſible, *as there are variations of theſe Signs*, and as many Negatives Impoſſible, *as there are Succeſſions of them without variation.*

For *by the Generation of Equations*, there are as many *Real* Roots, Affirmative or Negative, as there are Variations or Succeſſions of the Signs *in the Equations.* And *by the Rule we have produc'd*, there are as many *Impoſſible*, as variations of the Signs written under.

Therefore according as the variations of the Signs *under written* anſwer either to the variations or ſucceſſions of the Signs *written above*, ſo it is a plain Proof that the Impoſſible Roots lie conceal'd either among the Affirmative or Negative.

As in the Equation

$$\begin{array}{l} x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 \\ \;\;+ \quad\;\; + \quad\;\; - \quad\;\;\; + \quad\; + \;\; + \end{array}$$

$=0$, the varying under written Signs are $+\ -\ +$, to which above anſwer $-\ +\ -$, *viz.* of the Terms $-4x^4$, $+4x^3$ and $-2xx$.

Now the two variations of the under written Signs denote two *Impoſſible* ones, and the two variations of the above written Signs ſhew two *Affirmatives:* and conſequently the Impoſſibility lies conceal'd among the Affirmative, or the two aforeſaid Affirmatives are Impoſſible. And therefore ſince the Signs of the Equation propos'd (ſtanding in this order) $+ - + - - -$, do ſhew three Affirmative Roots, and two Negative, and two of the Affirmative ones are Impoſſible; it follows that the Roots of the Equation are to be ſo computed, that is, only one true or *real Affirmative* one, and *two Impoſſible* Affirmatives, and *two real Negative* ones.

Probl. 3. *To increaſe or leſſen the Roots of any Equation by any given Quantity.*

That is, any Equation being given, to form another in which the values of the Roots may be greater or leſſer than the values of the Roots of the Equation propos'd by any given Quantity.

Suppoſe another unknown Quantity + *or* − *a Quantity given, equal to the Root of the Equation given, into the place of which a new Expreſſion, ariſing from hence, is to be ſubſtituted throughout the whole Equation.*

For Example, Suppoſe an Equation $y^4 + 3y^3 - 15y^2 - 19y + 30 = 0$, and the Root y were to be increas'd by the Number 4.

Suppoſe $z = y + 4 \therefore z - 4 = y$, whence

$$\begin{array}{r} z^4 - 16z^3 + 96z^2 - 256z + 256 = y^4 \\ +3z^3 - 36z^2 + 144z - 192 = 3y^3 \\ -15z^2 + 120z - 240 = -15y^2 \\ -19z + 76 = -19y \\ +30 \\ \hline \end{array}$$

The Sum $z^4 - 13z^3 + 45z^2 - 11z - 70 = 0$; where the Roots (which before were 1, 3, −2, −5,) increas'd by the Number Four, become 5, 8, 2, −1, one of the Negatives being turn'd into an Affirmative. But that the nature of this Problem, as alſo the reaſon of the ſeveral Changes thence ariſing among the Roots of the Equation, may be better underſtood, let an Equation be form'd of any Roots whatever, *viz.* $y - a = 0$, $y + b = 0$, ſo that

$$\begin{matrix} y^2 - a \\ \quad + b \end{matrix}\, y - ab = 0,$$ is the Equation whoſe

Root (for Example) is to be increas'd by the given Quantity n. Therefore it will be $z = y + n$, $\therefore z - n = y$, and by ſubſtituting there ariſes

$$zz\begin{array}{l}-2n\\-a\\+b\end{array}z+nn+na-nb-ab=0.$$

But this Equation would be produc'd by multiplying $z-a-n$ into $z+b-n$, ſo that the values of z are $a+n$, and $-b+n$, and conſequently you have a new Equation, in which each of the Roots are greater than they were in the former by the Quantity n.

The *Affirmative* which before was a, is now $a+n$, and the *Negative* $-b$ is alſo $-b+n$, which indeed is greater than $-b$, by n, but yet it remains *Negative*, ſuppoſing that b exceeds n. The Expreſſion $-b+n$, is *a greater Quantity* than $-b$, tho' it muſt neceſſarily be expreſs'd *by a leſſer Number.* As $-4+3$ or -1, is ſomewhat greater than -4, yet 1 is a leſſer Number than 4. Whence to ſpeak in the *Style of Cartes* we ſhould ſay, *That the falſe Root is leſſen'd by the ſame Quantity*, as the *true one is rais'd.*

If we ſuppoſe $n=b$, then the Equation is turn'd into this, $zz-az-nz=0$, or $z-a-n=0$. That is, *If the Affirmative Root be augmented by a Quantity equal to any Negative Root, that Negative Root will diſappear, and conſequently the dimenſions of the Equation will be one degree leſſen'd.*

If $n>b$, then $-a+b-2n$ will have a Negative Sign, $nn-nb+an-ab$ will have an Affirmative one, ſo that the Negative Root by this means becomes Affirmative. That is, *If the Affirmative Root be augmented by a Quantity greater than one of the Negatives, that Negative will be turn'd into an*

Affirmative: and conſequently, if the increaſe of the Affirmative Root exceeds the greateſt Negative Root, all the Negative Roots will be chang'd into Affirmatives.

If $n<b$, then the Quantity $nn-nb+an-ab$ will be always a Negative one, but $-a+b-2n$ an Affirmative, if $b>a+2n$, but Negative if $b<a+2n$: *But in either Caſe, one Root will be Affirmative, and the other Negative, altho' in the firſt Caſe the Negative is greater, but in the latter, the Affirmative.*

From theſe few things, it may be eaſily gather'd, how it would be if the *Affirmative Root were to be* made leſs, or on the contrary the *Negative to be augmented*, by given Quantity; and that in any Equations whatever of higher Dimenſions: ſo that there is no need to inſiſt any longer on the Explication of them.

This Problem, *may be of Service for the following Uſes.*

Firſt, *To take away the ſecond Term of any given Equation.* or rather to change it into another Equation, which wants the ſecond Term.

Suppoſe $x^3-lxx+mmx-n^3=0$, and make $z=x+q$; for the *Sign*, as well as the Quantity *q*, ought to be diſcover'd, and therefore to be left as undetermin'd. Therefore it is $z\mp q=x$, which value being ſubſtituted into the place of *x*, there ariſes

z^3

$$\begin{array}{llll} z^3 \mp 3zzq & \pm 3zq^2 & \mp q^3 & \\ \quad -lzz & \mp 2zlq & + lq^2 & =0. \\ & + mmz & \mp mmq & \\ & & -n^3 & \end{array}$$

That the ſecond Term may be out, $3zzq$ and lzz ought to be equal, and ſhould have contrary Signs. But lzz by the *Hypotheſis* of the *Firſt Equation* has a negative Sign; therefore $3qzz$ or q it ſelf muſt be Affirmative, therefore it ought to be $z+q=x$. Moreover $3q$ ought to be equal to l, it is therefore $q=\frac{1}{3}l$, and therefore $z+\frac{1}{3}l=x$.

If in the *Equation given* it had been $+lxx$, then in the other it had been $+lzz$, and conſequently, $3zzq$ muſt have a Negative Sign, that is, $3q$ muſt be a Negative; whence it ought neceſſarily to be $z-q=x$, or $z-\frac{1}{3}l=x$.

In the former *Hypotheſis*, where $z+\frac{1}{3}l=x$, there had come forth this Equation,

$$\begin{array}{llll} z^3 * -\frac{1}{3}ll & & -\frac{2}{27}l^3 & =0, \\ \quad +m^2 & z & -n^3 & \\ & & +\frac{1}{3}lm^3 & \end{array}$$

In the latter where $z-\frac{1}{3}l=x$,

$$\begin{array}{llll} z^3 * -\frac{1}{3}ll & & +\frac{2}{27}l^3 & \\ \quad +m^2 & z & -\frac{1}{3}lm^2 & =0, \\ & & -n^3 & \end{array}$$

in both of which the ſecond Term is wanting.

To have a *General Rule*, we ought to propoſe an Equation in General Terms; *For Example*, $x^r - lx^{r-1} + mmx^{r-2}$, &c. $=0$. Then ſuppoſing $x=z\pm q$, it will be

$$\left.\begin{array}{l} z^r \pm rz^{r-1}q, \text{ \&c:} \\ -lz^{r-1}, \text{ \&c:} \end{array}\right\} = 0,$$

and that the ſecond Terms may deſtroy one another, $rz^{r-1}q$ ought to have an Affirmative Sign, ſince l (by the *Hypotheſis*) has a Negative one.

Further, it will be $l = rq$, or $q = \frac{l}{r}$, that is, *to the known Quantity of the ſecond Term, divided by the Number of the Dimenſions of the firſt. If therefore the Root of the given Equation be leſſen'd by this Quantity* (the Signs of the firſt and ſecond Terms being *contrary*) *or increas'd* (when they are *the ſame*) *the ſecond Term will be out.*

Now this taking away of the ſecond Term is uſeful for ſolving Quadratick adfected Equations. As if $yy - ay - bb = 0$, and it be $y - \frac{1}{2}a = x$, or $y = x + \frac{a}{2}$; then it will be

$$\left.\begin{array}{l} xx + ax + \frac{1}{4}aa \\ \quad - ax - \frac{1}{2}aa \\ \qquad - bb \end{array}\right\} = 0, \text{ or } xx = \tfrac{1}{4}aa + bb,$$

whence $x = \sqrt{\frac{1}{4}aa + bb}$, therefore $y = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$ But as the *ſecond* Term of an Equation may be taken away by the Solution of a *ſimple* Equation, ſo alſo the *third* Term may be taken away; but by the reſolution of an adfected Quadratic Equation.

As in the foreſaid Equation $x^3 - lxx + mmx - n^3 = 0$, where ſuppoſing $x = z + q$, there reſults

$$\begin{array}{l} z^3+3zzq+3zq^2+q^3 \\ \quad -lzz-2lzq-lq^2 \\ \qquad\qquad +m^2z+m^2q=0. \\ \qquad\qquad\qquad -n^3 \end{array}$$

That the *third* Term may vaniſh, it ought to be $3q^2-2lq+m^2=0$, that is, $q^2-\frac{2}{3}lq=-\frac{m^2}{3}$ or $q^2-\frac{1}{3}lq+\frac{1}{9}ll=\frac{1}{9}ll-\frac{mm}{3}$, therefore $q=\frac{1}{3}l\pm\sqrt{\frac{1}{9}ll-\frac{mm}{3}}$ Hence, ſince $z+q$ is written for x, there ought alſo to be written $z+\frac{1}{3}l\pm\sqrt{\frac{1}{9}ll-\frac{mm}{3}}$ for x in the Equation propos'd, that there may ariſe another Equation, in which the *third* Term may be wanting, and the *ſecond* not: Which may alſo be eaſily illuſtrated in Numbers.

The taking away of the other Terms (as of the *fourth*, the *fifth*, &c.) would require the reſolution of higher Equations; which therefore we omit as an uſeleſs piece of work.

Secondly, *By the help of this* Artifice, *by which the Values of the Roots are chang'd* (that is increaſed or leſſened) *deficient Equations may be fill'd up, ſo that no Term at all ſhall be wanting*: And beſides, that the Equation may *ariſe* to any given Number of Dimenſions.

For Example, If it be $x^3 * -rx+q=0$, ſuppoſe $x=y-n$, and the Powers of it in place of the Powers of x, which are found in the propos'd Equation; and a *Cubic* Equation will ariſe, in which there will be no Term wanting, if there be deſired a complete *Biquadratic* Equation, multiply the ſeveral Terms of the given Equation by x, whence it is $x^4 * -rx^2+qx=0$, and then ſubſtitute every where $y-a$ inſtead of x. So if it ſhould be $x^7****-bxx+cx-r=0$, and in the place of it another Equation of eight Dimenſions, which has all its Terms, be requir'd, let there be written $x^8****-bx^3+cx^2-rx=0$, and ſuppoſing $x=y-m$, let *a new Equation* be form'd out of the Value of the Root x, according as the Powers of the in-compleat Equation direct.

Thirdly, *This Operation is uſeful to the finding out Impoſſible Roots, which the fore-going Rule leaves ſometimes hidden and conceal'd.*

As if $z^3-3ddz-3d^3=0$, by trying the Impoſſible Roots according to the Rule, there will be found none at all. For writing (over the middle Terms) the Fractions which ſuit with the nature of the Equation

$$\underset{+\ \ +}{z^3\overset{\frac{1}{3}}{*}}\ \ \underset{+}{-3\overset{\frac{1}{3}}{d}dz}\ \ \underset{+}{-3d^3}=0,$$

'twill be $0>-3dd$, and $3d^4<0\times-3d^3$ $(=0)$ whence the under-written Signs ſtand in this order $++++$; and therefore you cannot conclude there is any Impoſſible Root.

But let the Root z be increa'd, that it may be *for example*, $y-q=z$,

$$\text{therefore }\begin{array}{l} y^3-3y^2q+3yq^2-q^3 \\ \phantom{y^3-3y^2q+{}}-3yd^2+3d^2q=0. \\ \phantom{y^3-3y^2q+3yq^2{}}-3d^3 \end{array}$$

Now it is plain that $\frac{1}{3}\times9q^2$ or $3q^2>3q^2-3d^2$, and therefore under the *ſecond* Term the Sign $+$ ought to be written. But comparing $\frac{9q^4-18q^2d^2+9d^4}{3}$, with the Quantity $3q^4-9q^2d^2+9d^3q$ (that is, the third part of the *Square* of the middle Term with the *Rectangle* of the Terms ſtanding on both ſides) or $9q^4 - 18q^2d^2 + 9d^4$ with $9q^4 - 27q^2d^2 + 27d^3q$, or (by *throwing out* on both ſides $9q^4$, *adding* $18q^2d^2$, and laſtly *dividing* by $9d^2$) comparing d^2 with $3dq - q^2$; if $d^2 > 3dq - qq$, then alſo under the third Term the Sign $+$ ſhould be written, and therefore there will be no Impoſſible Root. But if it be $d^2 < 3dq - qq$, in that Caſe we ought to write the Sign $-$ under the foreſaid Term, and then will preſently appear two Impoſſible Roots.

Therefore it is manifeſt, after what manner the *Increment* of the Root *ought to be taken*, to the end that the Impoſſible Roots, *which were before conceal'd, may now ſhew themſelves.* And indeed if it be $q = d$, the Equation will appear thus, $y^3 - 3y^2q * - d^3 = 0$, where at firſt ſight are diſcover'd two Impoſſible Roots.

Probl. 4. *To multiply or divide the Roots of a given Equation by any given Quantity.*

Suppoſe any other unknown Quantity *multiple* or *ſubmultiple* (*in any given ratio*) of the Root of the Equation propos'd, the new value of which Root ariſing from hence, is to be ſubſtituted in its place, and ſo there will be form'd the Equation Sought.

Suppoſe $z^3 - azz + nnz - p^3 = 0$, and ſuppoſe $v = rz$, therefore $z = \frac{v}{r}$, whence it will be $\frac{v^3}{r^3} - \frac{av^2}{r^2} + \frac{nnv}{r} - p^3 = 0$; that is multiplying all into r^3 $v^3 - rav^2 + rrnnv - p^3r^3 = 0$.

If the Quantity r be expounded by any *Integer* Number, this operation is the ſame as the Multiplication of the Root (z) by the ſame Number: But if r be a *Fraction*, then it is a diviſion of the Root z by the ſame Number. That is, the Root v is the *multiple* or *ſub-multiple* of z in the given *ratio* of r to 1. For Example, if $r=3$, or $z=\frac{v}{3}$, it will be $\frac{v^3}{27}-\frac{avv}{9}+\frac{nnv}{3}-p^3=0$, or multiplying all by 27, $v^3-3avv+9nnv-27b^3=0$. If $r=\frac{1}{3}$, or $z=3v$, it will be $27v^3-9avv+3nnv-p^3=0$, that is dividing all by 27, $v^3-\frac{avv}{3}+\frac{nnv}{9}-\frac{p^3}{27}=0$.

By which it is manifeſt, *That the known Quantities in the new Equation, are the known Quantities of the given Equation* reſpectively multiply'd *into the Powers of that Quantity, by which the Root is to be multiply'd or divided.*

This Operation will be of uſe ——

Firſt, *To make the known Quantity of any Term in an Equation, equal to any other given Quantity whatever.*

As if in the place of the former Equation $z^3-azz+nnz-p^3=0$, I ſhould demand *another*, in which the known Quantity of the ſecond Term ſhould not be a but d. Make $v=zq$ or $z=\frac{v}{q}$; where we are to determin the value of the Quantity q in the given Quantities a and d. Wherefore ſubſtituting $\frac{v}{q}$ in the place of z, it will be $\frac{v^3}{q^3}-\frac{av^2}{q^2}+\frac{nnv}{q}-p^3=0$, or $v^3-aqv^2+nnqqv-q^3p^3=0$. Now by the *Hypotheſis* the known Quantity of the ſecond term ought to be d, therefore $aq=d$, or $q=\frac{d}{a}$, and by putting $\frac{d}{a}$ for q in the laſt Equation, you will have $v^3-dv^2+\frac{nndd}{aa}v-\frac{p^3d^3}{n^3}=0$, an Equation in which the known Quantity of the ſecond Term is d, according as was deſir'd.

If in the place of the former an other Equation were requir'd, in which the third Term ſhould not be nn but ff; then making a new Equation (as before) it muſt be $nnqq=ff$, or $q=\frac{f}{n}$, whence will reſult the Equation $v^3-\frac{af}{n}v^2+ffv-\frac{f^3p^3}{n^3}=0$. in which the known Quantity of the third Term is f^2.

Secondly, *By this way Equations may be deliver'd (ſometimes) from Surd Quantities and Fractions, and therefore be brought to Expreſſions purely Rational and Integral.*

Suppoſe the Equation $x^3 - x^2\sqrt{p} + a^2x - \frac{d}{\sqrt{p}} = 0$, and it be $z = xq$, whence $\frac{z}{q} = x$, therefore $\frac{z^3}{q^3} - \frac{z^2\sqrt{p}}{q^2} + \frac{a^2z}{q} - \frac{d}{\sqrt{p}} = 0$, and $z^3 - qz^2\sqrt{p} + q^2a^2z - \frac{q^3d}{\sqrt{p}} = 0$.

If now the indeterminate Quantity q were $= \sqrt{p}$ or $p\sqrt{p}$ or $p^2\sqrt{p}$ or $p^3\sqrt{p}$ or $p^4\sqrt{p}$, *&c.* that is, $p^{\frac{1}{2}}$, $p^{\frac{3}{2}}$, $p^{\frac{5}{2}}$, $p^{\frac{7}{2}}$, $p^{\frac{9}{2}}$, &c. Then *all the Terms wou'd be rational*, that is, $\sqrt{p}$ would appear no more in the Equation. But this would not happen, if q were to be expounded by p. p^2. p^3. *&c.* for the Terms $q\sqrt{p}$ and $\frac{q^3}{\sqrt{p}}$ would always remain irrational. Let therefore $q = \sqrt{p}$, as the *moſt ſimple* Expreſſion, then it will be $z^3 - q^2z^2 + q^2a^2z - q^2d = 0$, the Equation deſir'd. Where we muſt take notice that (the known Quantity of the ſecond Term) $\sqrt{p}$ is multiply'd by q, and a^2 (the known Quantity of the third Term) by q^2, and $\frac{d}{\sqrt{p}}$ (the fourth Term) by q^3: For it is $q^2 = \sqrt{p} \times q$, $q^2a^2 = a^2 \times q^2$, $q^2d = \frac{d}{\sqrt{p}} \times q^3$.

In *numbers*, Suppoſe $x^3 - x^2\sqrt{3} + \frac{26}{27}x - \frac{8}{27\sqrt{3}} = 0$ Here is found $a^2 = \frac{26}{27}$, $d = \frac{8}{27}$, $\sqrt{p}$ or $q = \sqrt{3}$, which values being ſubſtituted in the former Equation, there ariſes $z^3 - 3zz + \frac{26}{9}z - \frac{8}{9} = 0$.

Let there be further propos'd an Equation in general Terms $z^3 - mzz + \frac{az}{d} - \frac{b}{d} = 0$, and let it be to be *freed from Fractions.* Imagine $v = rz$, and let the Quantity r neceſſary for this end be found, as the Quantity q was ſought before, for taking *away the Aſſymetry.* Therefore writing every where $\frac{v}{r}$ in the place of z, you will have (multiplying by r^3) $v^3 - mrv^2 + \frac{arrv}{d} - \frac{br^3}{d} = 0$. Theſe things being ſuppos'd, many Suppoſitions may be made, in all which *the Fractions will utterly diſ-appear.* As if the Quantity r be determin'd equal to *any Multiple* of the Quantity d, d and m in the mean while *being Integer Numbers.* Or if r or $rr = d$ and m be any Integer Number. Or laſtly, if $r = m = \sqrt{d}$, and m be expounded by an Integer Number.

But I undertake not to reckon up all the ways by which this may be perform'd. If it happen (*for Example*) that $m^2 = d$, $r = m$, and there will be this Equation $v^3 - mmv^2 + av - bm = 0$, *free from all Fractions.* In *Numbers,* if $z^3 - 3zz + \frac{26}{9}z - \frac{8}{9} = 0$, where $3 = m$, $9 = d$, $26 = a$, $8 = b$; ſuppoſe $v = 3z$, and at laſt will reſult $v^3 - 9vv + 26v - 24 = 0$. Where again appears the Multiplication of the known Quantities of the given Equation reſpectively, by the Terms of a *Series* of continual Proportionals.

Schol. By the Operations explain'd in this Problem we may ſee, how Equations, whoſe Roots *involve Surd or broken Quantities,* or are expreſs'd ei-

either by rational and integral Numbers; but perhaps too *great and difficult* ones, may be tranſmuted into other Equations, which are more eaſie, whoſe Roots may be *rational* or *integral*, or, leſs by any given Quantity than before.

Prob. 5. *To tranſmute the Roots of any Equation into their Reciprocals*; that is, to form an Equation, whoſe Roots are the Reciprocal of the Roots of the given Equation.

Suppoſe the Equation $x_3 - 2x^2 - 23x + 60 = 0$, and let $z = \frac{1}{x}$, $\therefore x = \frac{1}{z}$, whence $\frac{1}{z^3} - \frac{2}{z^2} - \frac{23}{z} + 60 = 0$, and $1 - 2z - 23zz + 60z^3 = 0$, or $z^3 - \frac{23}{60}zz - \frac{1}{30}z + \frac{1}{60} = 0$. The Roots of the given Equation were 4, 3, -5, therefore the Roots of the new Equation are $\frac{1}{4}$, $\frac{1}{3}$, $-\frac{1}{5}$, the Reciprocals of the former: Which is plain of it ſelf, for this Equation will be produc'd by multiplying $z - \frac{1}{4} = 0$, $z - \frac{1}{3} = 0$, $z + \frac{1}{5} = 0$, into one another. So that by this means the *greateſt* Root is chang'd into the *leaſt*, and the *leaſt* into the *greateſt*; and in this Example, the Root 3 into $\frac{1}{3}$ and 4 into $\frac{1}{4}$, and -5, being now $-\frac{1}{5}$ comes nearer to nought. By this Operation,

First, *Equations which want the ſecond Term, but have the third, are tranſmuted into others which have the ſecond Term, but want the third.*

As if $x^3 * + 3m^2x - n^3 = 0$, let it be $x = \frac{1}{z}$ whence $\frac{1}{z^3} + \frac{3m^2}{z} - n^3 = 0$, and $1 + 3m^2z^2 - n^3z^3 = 0$, or $z^3 - \frac{3m^2}{n^3}z^2 * - \frac{1}{n^3} = 0$.

Secondly, On the contrary, thoſe Equations which have the ſecond Term, but want the third, by this means are turn'd into others which want the ſecond and have the third.

Which is manifeſt by the laſt Equation $z^3 - \frac{3m^2}{n^3}z^2 * - \frac{1}{n^3}$, which (if you make $z = \frac{1}{x}$) will be $\frac{1}{x^3} - \frac{3m^2}{n^3x^2} - \frac{1}{n^3} = 0$, that is, $1 - \frac{3m^2x}{n^3} - \frac{x^3}{n^3} = 0$, or $x^3 * + 3m^2x - n^3 = 0$.

So alſo *Biquadratics* (in which are found the *third* and *fourth* Terms, but not the *ſecond*) are chang'd into other Equations, which have the *ſecond* and *third* Terms, but not the *fourth*. As if $x^4 * - px^2 + qx - r = 0$, you make $x = \frac{1}{z}$, wherefore $\frac{1}{z^4} - \frac{p}{z^2} + \frac{q}{z} - r = 0$, that is, $1 - pz^2 + qz^3 - rz^4 = 0$, or $z^4 - \frac{q}{r}z^3 + \frac{p}{q}z^2 * - \frac{1}{r} = 0$.

Thirdly, *By an Operation of this ſort may deficient Equations be completed.* For writing the former Equation $x^4 * - px^2 + qx - r = 0$, and make $x =$

1

$\frac{1}{z^{\frac{1}{2}}}$, wherefore $\frac{1}{zz}-\frac{p}{z}+\frac{q}{z^{\frac{1}{2}}}-r=0$, that is, $1-pz+qz^{\frac{3}{2}}-rz^{2}=0$, therefore $qz^{\frac{3}{2}}=pz+rz^{2}-1$, whence by ſquaring each part it will be $qqz^{3}=ppz^{2}+rrz^{4}+2prz^{3}-2rz^{2}-2pz+1$, which is a *Biquadratic* Equation which has all its Terms. The ſame thing may be perform'd in others, *and with other aſſum'd Powers of the unknown Quantity*, though not *with the ſame facility.*

As if $y^{3} * -qy+s=0$, make $y=\frac{1}{z^{\frac{1}{3}}}$, wherefore $\frac{1}{z}-\frac{q}{z^{\frac{1}{3}}}+s=0$, and therefore $1-qz^{\frac{2}{3}}+zs=0$. whence $1+zs=qz^{\frac{2}{3}}$, and by multiplying *Cubically* it will be $1+3zs+3zzss+z^{3}s^{3}=q^{3}z^{2}$.

Prob. 6. *To expreſs a given Equation which involves irrational Quantities in pure rational ones*, and that in the Terms of that unknown Quantity, which is firſt found in the Equation.

Suppoſe $\sqrt[3]{bbx}-\sqrt{dx}+c=0$. Suppoſe $\sqrt[3]{bbx}=y$, $\sqrt{dx}=z$, whence comes the following Equation, $y-z+c=0$. Now here are two different unknown Quantities, of which we are to compoſe an Equation, which contains only the *rational* Powers of x. To this end, take the two Values of the unknown Quantity x, which reſult from the fictitious Equations, *viz.* $x=\frac{y^{3}}{bb}=\frac{zz}{d}$, $\therefore y^{3}=$

bbzz

$\frac{bbzz}{d}$, ∴ $\sqrt[3]{\frac{bbzz}{d}}-z+c=0$, or $\sqrt[3]{\frac{bbzz}{d}}=z-c$, ∵ multiplying Cubically $\frac{bbzz}{d}=z^3-3z^2c+3zc^2-c^3$, and becauſe $\sqrt{dx}=z$, making a ſubſtitution it will be $bbx=dx\ \sqrt{dx}-3dxc+3c^2\sqrt{dx}-c_3$, that is, $\frac{bbx+3dxc+c3}{dx+3c^2}=\sqrt{dx}$, and by taking the Squares on both ſides, it will be $\overline{bbx+3dxc+c3}|^2=dx\times\overline{dx+3c^2}|^2$, the *rational* Equation which was to be found.

Schol. If an Equation be given which involves more irrational Terms, then alſo there will be more unknown Quantities of a *different Name.* But becauſe as many Equations will be given, as unknown Quantities, *they may be thrown out by degrees*, and ſo (according to this Example) may be had an Equation free from all Surd Quantities, and expreſt alſo in the Terms of the unknown Quantity firſt given.

Prob. 7. *To give the Analytical Reſolution of Cubic adfected Equations, that is to change them* (when it is poſſible) *into ſimple Cubics.*

The ſecond Term of any Equation (and therefore of any Cubical) may be taken away, by *what went before.* Therefore taking away the ſecond Term, proceed thus. Suppoſe $v^3+pv-q=0$, and let $v=z-x$, which value being ſubſtituted, you will have $z^3-3zzx+3zxx-x^3+pz-px-q=0$, that is $z^3-x^3-\overline{z+x}\times3zx=q-pz+px$. Make $z^3-x^3=q$, ∵ $p\times\overline{x-z}=3zx\times\overline{x-z}$ and by conſequence

$p=$

$p=3zx$, or $z=\frac{p}{3x}$, and therefore ſince (by the *Hypotheſis*) $z^3-x^3=q$, you will alſo have $q=\frac{p^3}{27x^3}-x^3$, or (reducing and ordering the Equation) $x^6+qx^3-\frac{1}{27}p^3=0$, and extracting the Square Root, there riſes $x^3=\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}-\frac{q}{2}$ which is a ſimple Cubic Equation.

Whence $x=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}$, & z

$$\left(=\frac{\frac{1}{3}p}{x}\right)=\frac{\frac{1}{3}p}{\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}}, \text{ \&}$$

$v\ (=z-x)=$

$$\frac{\frac{1}{3}p}{\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}}-\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}.$$

But if in the Equation $q=z^3-x^3$, you ſubſtitute the found value of x^3, and extract the Cube Root, you will have $z=\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}$, and ſince we ſuppoſe $v=z-x$, the value of x being ſubſtracted from thence, there we ſhall have the Expreſſion of the Root v according to the Rules of *Cardan*.

But the firſt way of expreſſing the Root is more eaſie and ſimple, for this reaſon, becauſe we need extract the Cube Root but once (which by the Rule of *Cardan* ſhould have been extracted twice) as will be plain to any one, who conſiders both the forms.

If the Equation propos'd had been $v^3 - pv - q = 0$, then by making $v = x - z$, would ariſe $3zx = -p$, or $z = \frac{-p}{3x}$, whence the Quantity $\frac{1}{27}p^3$ would have a Negative Sign. And therefore $\frac{1}{27}p^3$ being greater than $\frac{1}{4}q^2$, in that caſe the Expreſſion of x would be impoſſible; and conſequently the tranſmutation of the propos'd Equation, into a *ſimple* Cubic, would be impoſſible.

The ſame may be done another way, thus. Let there be given the Equation $x^3 + 3mmx - n^3 = 0$, and in the place of x write $\frac{zz - mm}{z}$ thro' the whole Equation, by which means will be found

$$\frac{z^6 - 3mmz^4 + 3m^4zz - m^6}{z^3} + \frac{3mmz^2 - 3m^4}{z} - n^3 = 0,$$

and reducing $z^6 - n^3z^3 - m^6 = 0$, or $z^3 = \frac{1}{2}n^3 + \sqrt{\frac{1}{4}n^6 + m^6}$ which is a ſimple Cubic Equation and always poſſible.

The Equation remaining, as before, if we ſuppoſe $x = \frac{mm - zz}{z}$, will ariſe alſo a ſimple Cubic Equation, and always poſſible.

But if the Equation be $x^3 - 3mmx - n^3 = 0$, then ſuppoſing $x = \frac{zz + mm}{z}$, there will be produc'd an Equation, which will be poſſible only when the Quantity m^6 is not found greater than $\frac{1}{4}n^6$, for

otherwiſe

wiſe the Root of a Negative Square would be to be extracted, from which it would be concluded that the tranſmutation of the given Equation into a ſimple Cubic one, would be altogether Impoſſible.

Prob. 9. *To give the Solution of Biquadratic Equations, or to ſhew how they may be divided into two Quadratic adfected ones.*

As *Cubic* Equations, which are without the ſecond Term, may firſt be reduc'd to a Quadratic adfected Equation, and then ſometimes to a ſimple Cubic; ſo likewiſe may *Biquadratics*, which alſo want the ſecond Term, by the means of a *Cubic* adfected Equation, be eaſily divided into two *Quadratic* adfected ones, and ſo their Roots may be found.

Let us take that Equation, which *Cartes* himſelf (*Geom. B.* 3. *Pag.* 79.) has made uſe of as an Example of this Rule, *viz.* $x^4 * . pxx . qx . r = 0$; or by diſtinguiſhing it by Signs at Pleaſure, let $x^4 * - px^2 - qx + r = 0$. He ſays that in its place you may write the two *Quadratic* Equations $xx + yx + \frac{1}{2}yy - \frac{1}{2}p + \frac{q}{2y} = 0$, and $xx - yx + \frac{1}{2}yy - \frac{1}{2}p - \frac{q}{2y} = 0$, that is, having found the value of the Quantity y, by the help of this *Cubic* Equation

$$\begin{array}{l} y^6 - 2py^4 + pp \\ \qquad\qquad\quad - 4r\,yy - qq = 0. \end{array}$$

To explain the foundation of this Rule, imagin two *Quadratic* adfected Equations, by the multiplication of which there may come forth a *Biquadratic* Equation of the ſame form with that which is propos'd. Let them be $xx + yx + z = 0$, and $xx - yx + v = 0$, where the Term yx has the poſitive Sign in one Equation, but the Negative in the other, to the end that the Term yx^3 may be loſt in the Equation *which reſults*, becauſe x^3 is wanting in the given Equation. Moreover, the Quantities, y, z, and v are to be found ſeverally, that is z and v in the Terms of y, and known Quantities, and y (if ſo be it is poſſible) by the Cubic Equation aforeſaid.

$$\begin{array}{rllll} \text{Now multipl.} & xx + yx + z = 0 & & \\ \text{by} & xx - yx + v = 0 & & \\ \text{prod.} & x^4 + z & -zy & \\ & - yy\,xx & + vy\; x & + vz = 0. \\ & + v. & & \end{array}$$

Compare this with the given Equation $x^4 - pxx - qx + r = 0$, that is, let thoſe Terms be equal'd, in which the ſame Powers of the Quantity x are found. Whence firſt, $z - yy + v = -p$, $\therefore z + v = yy - p$; 2ly, $vy - zy = -q$, $\therefore v - z = -\frac{q}{y}$; 3ly, you will have $vz = r$.

Now from hence to find z, you have $\left\{ \begin{array}{l} v = yy - p - z \\ v = z - \dfrac{q}{y} \end{array} \right.$

$\therefore yy - p - z = z - \frac{q}{y}$, and reducing $z = \frac{y^3 - py + q}{2y}$ whence becauſe $v = yy - p - z$; you will have $v =$

$yy - p - \frac{y^3 + py - q}{2y} = \frac{y^3 - py - q}{2y}$. So that z and v, are $\frac{1}{2}yy - \frac{1}{2}p + \frac{q}{2y}$ and $\frac{1}{2}yy - \frac{1}{2}p - \frac{q}{2y}$, and by conſequence the Quadratic Equations, which we had at firſt form'd, will return to thoſe very Equations which *Cartes* had deſcrib'd.

But this is nothing, unleſs we diſcover an Equation at the ſame time to find out the value of the Quantity y. And to obtain ſuch a one, multiply the values of the Quantities v and z into one another, and make the Product equal to r, (becauſe by what has been ſaid before, it is $vz = r$) that is,

$$\frac{y^3 - py + q}{2y} \times \frac{y^3 - py - q}{2y} = \frac{y^6 - 2py^4 + ppy^2 - q^2}{4yy} = r,$$

and the Equation being diſpos'd in order, there will ariſe

$$y^6 - 2py^4 \begin{array}{l} +pp \\ -4r \end{array} yy - qq = 0,$$ which is the very Cubic Equation which *Cartes* had order'd ſhould be written.

But adding an Example we ſhall illuſtrate this *Inveſtigation* in *Numbers*, that the Young Student may more clearly perceive the method of ſo uſeful and pleaſant a Practiſe.

Suppoſe $x^4 - 17xx - 20x - 6 = 0$, to be the Equation propos'd.

$$\text{The two } \textit{feign'd} \text{ Quadratic Equations} \left\} \begin{array}{l} xx + yx - z = 0 \\ xx - yx + v = 0 \end{array} \right.$$

$$\text{Equation reſulting from the multiplication of theſe} \left\} x^4 \begin{array}{l} -z \\ +v \\ -yy \end{array} xx \begin{array}{l} +zy \\ \\ +vy \end{array} x - vz = 0 \right.$$

Compariſon between the Terms of two Biquad. Equat.

$$v-z-yy=-p=-17,\ \therefore v-z=yy-17.$$
$$yv+zy=-q=-20,\ \therefore v+z=-\frac{20}{y}.$$
$$-vz=-r=-6.$$

The two values of the Quantity *v* equal'd to one another

$$v=yy-17+z$$
$$v=-\frac{20}{y}-z$$
$$\therefore yy-17+z=-\frac{20}{y}-z$$

The finding out the Quantity *z* by the two values of *v*

$$z=\frac{17y-y^3-20}{2y}$$

The Quantity *v* found by *z*, by either Equat.

$$v=yy-17+z=$$
$$yy-17+\frac{17y-y^3-20}{2y}=$$
$$\frac{y^3-17y-20}{2y}$$
$$\text{or } v=-\frac{20}{y}-z=$$
$$\frac{-20}{y}-\frac{17y+y^3+20}{2y}=$$
$$\frac{y^3-17y-20}{2y},\ \text{as before.}$$

The Product $-vz$ equal'd to -6, or $-r$, and the Equation ariſing from thence. $\Big\}$ $\frac{y^3-17y-20}{2y} \times \frac{17y-y^3-20}{2y} = 6$, ∴ reducing and changing the Signs, $y^6-34y^4+313y^2-400=0$

Having found the value of yy by this Cubic Equation, *viz.* $yy=16$, or $y=4$, the two Quadratics which were at firſt ſuppos'd are thus determin'd. You will have z $\left(=\frac{17y-y^3-20}{2y}=\frac{17}{2}-8-\frac{20}{8}=\frac{1}{2}-\frac{5}{2}\right)=-2$, and therefore $xx+yx-z=0=xx+4x+2$. In like manner v $\left(=\frac{y^3-17y-20}{2y}=8-\frac{17}{2}-\frac{5}{2}=-\frac{1}{2}-\frac{5}{2}\right)=-3$, and therefore $xx-yx+v=0=xx-4x-3$.

Now the Roots of the Equation $x^2+4x+2=0$, are $\sqrt{2}-2$, & $-2-\sqrt{2}$, for thoſe being join'd together with x, and multiply'd into one another will bring forth the Equation $x^2+4x+2=0$. Likewiſe the Roots of the Equation $xx-4x-3=0$, are $\sqrt{7}+2$ & $2-\sqrt{7}$.

Therefore theſe will be the four Roots of the given Equation $x^4-17x^2-20x-6=0$, ſince it is produc'd by multiplying x^2+4x+2 by $xx-4x-3$.

And according to the Example of this Rule we are to proceed in other Caſes which may occur.

Schol. (To *the two proceeding Problems*). When the Root of a Cubic Equation *transform'd*, is *poſſible*, two of the Roots of the Equation firſt *given*, are impoſſible. But *that being Impoſſible, all their three Roots of this Equation are Poſſible.* Further, if the Roots of the given Biquadratic Equation are *poſſible*, then three Roots of the Cubic Equation (which is generated by the Method we have ſhewn) will be alſo poſſible, and therefore it can never be chang'd into a ſimple Cubic Equation, or its Root be obtain'd by ſuch a way.

Prob. 10. *To aſſign the Limits within which the Roots of Algebraic Equations conſiſt.*

It will not be foreign to our purpoſe to premiſe this ſhort Obſervation before we enter upon the Problem.

If the Terms of any Equation be reſpectively multiply'd by the Indices of the Powers of the unknown Root, and this be done continually, and a Quantity, greater than the greateſt Affirmative Root, be ſubſtituted in the place of the Root, as well in the Equation as in all theſe Products: I ſay, that at laſt an Affirmative Quantity will remain every where.

Suppoſe a *Quadratic* Equation affected with Signs any how,

$$xx \begin{array}{l} -a \\ +b \end{array} x - ab = 0.$$

Multiply the ſeveral Terms by the Exponents of the Powers of the Root x, after this manner,

$$\begin{array}{ccc} x^2 & -ax & -ab \\ & +b & \\ 2 & 1 & 0 \end{array}$$

the Product will be

$$2x^2 \begin{array}{l} -a \\ +b \end{array} x, \text{ or } 2x - a + b.$$

Now ſubſtitute a Quantity, as $a+1$, which exceeds the Affirmative Root a, in the place of x thro' the whole Equation, and the *Product*. And then it will be $aa+2a-aa-a+ba+b-ba=$ an *Affirmative* Quantity, and at the ſame time $2a+2-a+b$ equal to an Affirmative Quantity.

Suppoſe alſo the Cubic Equation

$$x^3\begin{array}{l}-a\\-b\\+c\end{array}x^2\begin{array}{l}+ab\\-ac\\-bc\end{array}x+abc=0.$$

If the Terms are multiply'd by the *Indices* of the Powers of x, that is by the Progreſſion 3. 2. 1. 0, there will ariſe *firſt*,

$$3x^2\begin{array}{l}-2a\\-2b\\+2c\end{array}x\begin{array}{l}+ab\\-ac\\-bc\end{array};$$

and the Terms of it being alſo multiply'd into the Terms of the Progreſſion 2. 1. 0. will ariſe *ſecondly* $6x-2a-2b+2c$. Let a be ſuppos'd the *greateſt of all the Affirmative Roots*; and writing $a+1$ for x through the whole Equation, and through the two Products ariſing from thence, you will have

1. $a^3+3a^2+3a+1-a^3-2a^2-a-ba^2-2ba-b+ca^2+2ca+c+ba^2+ba-ca^2-ca-bca-bc+abc$ $=$ *an Affirmative Quantity*, that is, throwing out the Members which deſtroy one another, you will have $a^2+2a+1+c-b-ba+ac-bc=$ an *Affirmative* Quantity, which is plain, ſince by the Hy-

potheſis $a > b$, $\therefore ac > bc$, $a^2 > ba$, therefore the whole Expreſſion is Affirmative.

2. In like manner, $3a^2 + 6a + 3 - 2a^2 - 2a - 2ba - 2b + 2ca + 2c + ab - ac - bc$, or $a^2 + 4a + 3 - ab - 2b + 2c + ac - bc =$ an *Affirmative* Quantity.

3. $6a + 6 - 2a - 2b - 2c$, that is, $4a + 2c + 6 - 2b =$ an *Affirmative* Quantity.

So that it is plain, that an *Affirmative Quantity* will ariſe every where, as well in the *Equation it ſelf*, as in the *Products* generated after the manner preſcrib'd.

Whence, on the contrary, that Quantity which being ſubſtituted in the place of the unknown Root, makes all theſe Sums Affirmative, will *neceſſarily exceed the greateſt Affirmative Root*.

And the *leaſt* of all thoſe Quantities which can perform this, is one of the *limits* of the Equation which were ſought, *viz.* it is the *limit* on the part of the Affirmative Roots.

And that which brings this to paſs, when the Signs are chang'd in the alternate Places through all theſe Sums, is another *limit*, or that which is on the part of the *Negative* Roots.

Theſe things being ſuppos'd, let us bring an Equation for an Example.

$$\begin{array}{ll} & x^3 - 17x^2 + 54x - 350 = 0 \;\; * \\ \text{Mult.} & \underline{3 \qquad 2 \qquad 1 \qquad 0} \\ \text{Prod.} & \quad 3x^3 - 34x^2 + 54x \end{array}$$

or dividing by x, there is left

$$\begin{array}{lccc} & 3xx & -34x & +54 \; * \\ \text{Mult.} & 2 & 1 & 0 \\ \hline \text{Prod.} & 6xx & -34x & \end{array}$$

or dividing alſo by x

$$6x-34 \; *$$

Now we are to find out a Number which in all the Sums (mark'd with an *Aſterisk*) being written in the place of x, may leave them *Affirmative*, that is, will cauſe the Reſult to be Affirmative. And indeed by trying the Numbers from 1 to 14 it is found that the thing does not ſucceed.

Nor will it ſucceed if 14 be put for x.

For it will be $6x-34=84-34=+50$
$3x^2-34x+54=588-476+54=+116$
$x^3-17x^2+54x-350=2744-3332+756-350$ $=-182$, whence becauſe this laſt comes out Negative, therefore the *limit* is greater than 14.

Write therefore 15 for x, and then it will be

$6x-34=90-34=+56$, and
$3xx-34x+54=675-510+54=+219$

$x_3 - 17x^2 + 54x - 350 =$
$3375 - 3825 + 810 - 350 = +10.$

Wherefore ſince, ſuppoſing $x = 15$, an Affirmative Number will ariſe in all the aforeſaid Sums, the Number 15 will be the *limit Sought*, that is, *greater* than the *greateſt* Affirmative Root.

But we are to take notice, that tho' there appear to be three Affirmative Roots in this Equation, yet two of them are *Impoſſible*, and therefore the Equation cannot be expounded but by one Root, and that an Affirmative one.

CHAP.

CHAP. V.

Of the Nature of Problems in general, and of their various Diſtinctions.

THe diſtinction of Problems into *Determinate* and *Indeterminate*, is in the firſt place to be conſider'd (as being the moſt Univerſal).

Of the former ſort are thoſe which admit of but one Solution, or a certain *Determinate* Number of Solutions.

Indeterminate Problems are thoſe which are capable of an Indefinite Number of Solutions, which will anſwer all their Conditions.

Frequent Examples of ſuch occur both in *Arithmetick* and *Geometry*.

As if an *Integer Square Number* were inquir'd, which may be divided without a Fraction, by 3, 4 and 5.

Or if a *Rectangle* Triangle be requir'd, whoſe three ſides are *Rational*, that is (compar'd with one another) as *Number* to *Number*.

In Queſtions of this ſort, we may always ſupply the deficient determinations by Quantities taken at Pleaſure in ſuch a manner as will beſt ſuit with the proceſs of the Work. But we muſt take care in aſſumptions not to go beyond the Bounds which the nature of the Queſtion requires. A Queſtion (tho' *indetermin'd)* may have conditions which may *reſtrain* and confine it to certain limits.

We ought well to diſtinguiſh the Conditions which *reſtrain* only, from thoſe which fully and abſolutely *determine*.

Furthermore, it will be of Uſe to conſider, that ſometimes the Solutions of theſe Queſtions are ſo perfect and general, as to comprehend all poſſible Caſes. But ſometimes more narrow and imperfect, for this reaſon, becauſe tho' they exhibit *innumerable* Anſwers, yet not *all* which can be exhibited, or perhaps not thoſe which are moſt to to the purpoſe. Wherefore we are to diſtinguiſh between a *true* Solution and a *perfect* one.

But to deſcend from theſe general Things to others more particular and uſeful, we ſhall explain the moſt common and noted Senſe of an Indeterminate Problem among *Geometricians*.

In *Geometry*, an Indeterminate Problem is, when a Point Sought, by which all the conditions of the Problem are fully ſatisfy'd, may be indifferently taken through a certain extenſion, of whatever Dimenſions that extenſion be; whether a *Line*, a *Superficies*, or a *Solid*.

And *this Extenſion*, the ſeveral Points of which being taken at Pleaſure, will ſo anſwer the conditions of the Problem, is commonly call'd a *Place*, and the Problems of that kind *Local*, that is, ſuch as relate to *Places*.

A *Place* ariſes from the want of Matter or conditions in the Problem, by which it comes to paſs that as many Equations are not ſupply'd, as there are requir'd unknown Quantities. As if (for *Example)* we ſhould ſuppoſe two unknown Quantities in any Problem, and *having anſwer'd all the Conditions of it*, there is Matter afforded only for the *finding out of one of them*; then one of the unknown

known Quantities would be neceſſarily expreſt in the Terms of *the other* and *given Quantities.* And ſo that unknown Quantity to which no Equation anſwers, may be taken at Pleaſure; and ſo as many different values for the *other* unknown may be found out, as we imagine the other taken at Liberty to have. And ſince theſe values may be Infinite, there will be alſo an Infinite Number of Points by which the conditions of the Problem may be accurately anſwer'd; that is, for the Solution of ſuch a Problem, there will come out a certain Extenſion, which *Geometricians* commonly call a *Place.*

A *Locus* or *Place* in General is *Three-fold*, that is, *Locus ad Lineam*, *ad Superficiem*, *& ad Solidum.*

Locus ad Lineam is, when the Point which ſatiſfies the Problem, is found in a *Line*, whether right or *curve*; and that by reaſon of the defect of one Condition only, which is wanting to render the Problem *wholly determinate.*

Again, *Locus ad Lineam* may be *Four-fold*, viz. *Simple*, *Plane*, *Solid*, *Sur-Solid*, that is, *more than Solid.*

A *Simple Place*, is when the Point which ſatiſfies the Problem, is found in a *Right Line.*

A *Plane Place* when the ſame is found in the *circumference of a Circle.*

A *Solid Place* when it is found in any other *Conic ſection* beſides a Circle, *viz.* in an *Ellipſis*, *Hyperbola*, or *Parabola.*

A *Surd-Solid* Place, when in a curve of a *ſuperior Gender*, or *more compounded* than *Conicſections* are.

Some indeed reckon only a Plane, Solid and Sur-ſolid Place, comprehending under a Plane

Place a *right Line* as well as the *Circumference of a Circle.* But this difference is rather *verbal* than *real*, for doubtleſs a Plane-Place may be either extended to a *right* Line, or reſtrain'd to a Circle only, without any miſtake or danger.

Further, as to the *Equations*, by which *theſe Places* are expreſt, where the Place is *Simple* there is neither any *Product* or *Power* of the unknown Quantities in the Equation above the *firſt* or *ſimple*

For a *Plane* and *Solid* Place, there will always be found in the Equation either the Square of *one* of the unknown Quantities (as in the *Parabola*) or the *Squares of both* (as in the *Ellipſis*, the *Circle* and *Hyperbola*) or perhaps only a *Rectangle* under them (as it happens in Equations which belong to the *exterior Hyperbola)* but in no Caſe (that is, *after reduction to the moſt ſimple Terms*, which muſt always be underſtood) will it be found beyond a *Square*, or (which is equivalent to a Square) a *Plane* under the two unknown Quantities.

When the *Place* is *Sur-ſolid*, the Products or Powers of the unknown Quantities will ariſe up to 3 or 4, to 5 or 6, to 7 or 8, *&c. Dimenſions*, according as the *Line*, which anſwers the Problem, is either of the *ſecond*, *third*, *fourth*, *&c. Gender.*

For we are to take notice, that Curve Lines are uſually diſtinguiſh'd into certain *Genders*, by Equations which expreſs the Relations which *their ſeveral Points* have to *the Points of a right Line.* So that when this Equation does not aſcend beyond two Dimenſions, the Curve, whoſe Nature is expreſt by it, is ſaid to be of the *firſt Gender*, when it aſcends to 3 or 4, of the *ſecond*; when to 5 or 6, of the *third*, and ſo on *ad infinitum.* So much concerning *Locus ad Lineam.*

Locus ad Superficiem, is when there being two Conditions wanting to determine any Point which ſatisfies a Problem, that Point may be taken throughout the extenſion of ſome *Superficies*, *viz.* either Plane or Curve.

But when *three Conditions* are wanting to the determination of the Point ſought, the *Place* in which it is found is a *Solid*; and which may be concluded either under *Plane*, or *Curve*, or *mix'd Sxperficies*, and thoſe either *determin'd* or *indefinitely* extended.

And hence it appears that Equations (to which Problems are brought where the *Place* is either in a *Superficies*, or a *Solid*) ought to be *Identical*; that is, ſuch that having perform'd the Conditions, and duly reduc'd them, one and the *ſame Quantity will be found on both ſides of the Equation*; and therefore that Queſtions of this nature are really Propoſitions or *Theorems.*

For ſince *when the Locus is in a Line* one Condition only is wanting, from the Nature of the Equation reſulting one of the indeterminate Quantities will neceſſarily enter the expreſſion of the other, that is, by the infinite Values of the one taken at pleaſure, you may aſſign as many to the other. But when two Conditions are wanting (as in a *Locus ad Superficiem*) one of the unknown Quantities is never to be expreſt in the Terms of the other: for otherwiſe *one* Condition *only* would be wanting to render the Queſtion *determinate*, and ſo it would be not *Locus ad Superficiem*, but *ad Lineam.*

Nor, in like manner, when *three Conditions* are wanting, will the expreſſion of one unknown Quantity be obtain'd in the Terms of the other;

for if this could be brought to paſs, only two Conditions would be wanting, and then the *Place* would not be *ad Solidum*, but *ad Superficiem.* It is neceſſary therefore, that the Terms in theſe Equations be ſo diſpos'd, that the *ſame Quantity* (whatever it be) *may be found at laſt on both ſides of the Equation*, and nothing beſides.

For every Equation is ſuch, that an unknown Quantity *may* either *be*, or *not be* expreſt by it; and conſequently one of theſe will happen in the Equations for a *Locus ad Superficiem*, or *ad Solidum*.

But it is againſt the nature of theſe Places, that one of the indeterminate Quantities ſhould be expreſs'd by another, becauſe in that caſe fewer Conditions would be wanting, than the nature of them will ſuffer to be wanting. And therefore they are no way to be expreſs'd by the Equations which reſult. For whatever is expreſt, it neceſſarily muſt be expreſt either by *given* or *indeterminate* Quantities; but *given* or *determinate* expreſſions are quite foreign to this buſineſs. Therefore the Quantities will ſo utterly deſtroy one another on both ſides, that at laſt nothing will remain, except *the ſame thing equal to itſelf*.

But we ſhall illuſtrate all theſe things by fit Examples, when we ſhall come to diſcourſe in another place at large, *concerning the Invention and Conſtruction of Places*, and *ſome of their Uſes, which are not very common.*

Wherefore having done with *Indeterminate* Problems, we come now to ſee how many kinds of *determinate* Problems there are.

Now, Geometricians commonly divide them into four Claſſes, like the former, *viz.* into *Simple*, *Plane*, *Solid*, *Sur-ſolid* or *Linear*, as ſome call them.

That is a *Simple* Problem, in which, after an Equation has undergone the laſt Reductions, the unknown Quantity does *not aſcend beyond one Dimenſion*; and therefore only *right Lines* are neceſſary to the *couſtruction* of it.

A *Plane* Problem is when (an Equation being brought to the moſt ſimple Form) the unknown Quantity aſcends no higher than a *Square* or two Dimenſions; and the *Conſtruction* (that is, the *Geometrical Solution*) may be perform'd by making uſe of *right Lines* and *Circles*.

But a Problem is *Solid*, when having finiſh'd the Reduction, the unknown Quantity aſcends to *three* or *four* Dimenſions, and therefore cannot be found, or the Equation conſtructed, but by making uſe of a *Circle* and ſome *Conic-ſection*.

A Problem is *Linear* or *Sur-ſolid*, when the unknown Quantity ariſes to 5 or 6, 7 or 8, 9 or 10 Dimenſions, *&c.* the Equation not being to be reduced to lower Dimenſions; the Conſtruction of which (beſides a Circle) neceſſarily requires a Curve of the *ſecond*, *third*, *fourth* Gender, *&c.*

Hence a *ſimple* Problem has but *one Solution*, a *Plane* not more than *two*, nor a *Solid* more than *four*, nor a *Sur-ſolid*, by the help of a Circle and a Curve-Line of the ſecond Gender, *&c.* more than *ſix*.

For two *right Lines* cut one another only in one Point. A *right Line* and a *Circle*, in two only. A *Circle* and a *Conic Section*, in no more than *four*. Nor a *Circle* and a *Curve* of the *ſecond Gender*, in more than ſix, *&c.*

And from hence we come to the Conſtruction of ſolid Problems, which were added to this Book from *Cartes's* Geometry, by that *Excellent* and *every way Learned Mathematician*

Mr. *Edward Paget*.

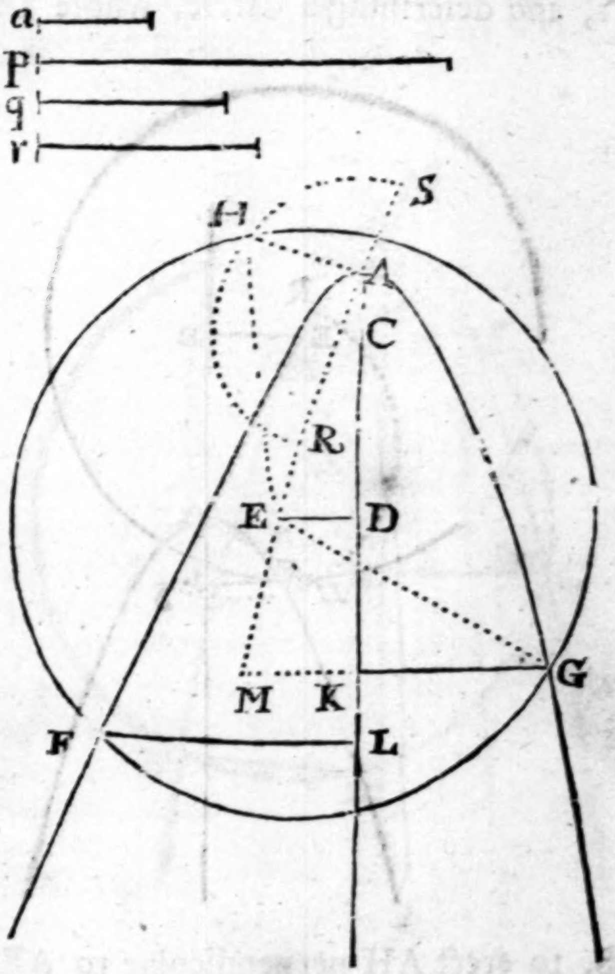
A general way of conſtructing all Solid Problems reduc'd to an Equation of three or four Dimenſions.

(From the Geometry of CARTES, *B.*III.*p.*85)

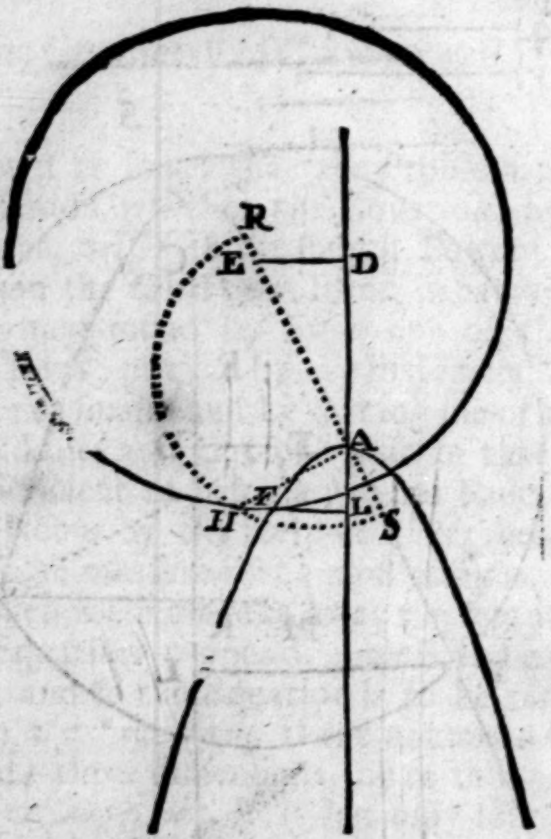
AFter it is found that the Problem propos'd is Solid: whether the Equation, by which it is Sought, ariſes to the fourth Power; or no higher than the third: its Root, whatever it be, may be always found by ſome one of the three Conic Sections; or even by any little part of them, tho' never ſo ſmall, and by making uſe of nothing but right Lines and Circles. But in this place it will be ſufficient to bring a general Rule how to find all Roots by the help of a Parabola, ſince that in ſome meaſure is the moſt Simple.

Firſt then we muſt take away the ſecond Term of the Equation propos'd, unleſs it be already wanting, and ſo the Equation is to be reduc'd to this form $z^3 = * apz . aaq .$ if the unknown Quantity has only three dimenſions, or to this $z^4 = * apzz \; aagz . a^3r$, if it has only four dimenſions; or by taking *a* for unity, to theſe $z^3 = * pz . q . \; z^4 = * pzz . qz \; r .$

Then by ſuppoſing the Parabola FAG to be already deſcrib'd, and its axis to be ACDKL, and its *latus rectum a* or 1, of which AC is the half, and laſtly the point C to be within this Parabola whoſe vertex is A. We are to make CD$=\frac{1}{2}p$, and to take [illegible] in the line AC, continued towards C, if in the Equation you have $+p$, but towards the other ſide if you have $-p$. Moreover from the Point D, or from the Point C, if the Quantity p be not in, by erecting DE$=\frac{1}{2}q$, perpendicular to the axis we are to deſcribe the Circle

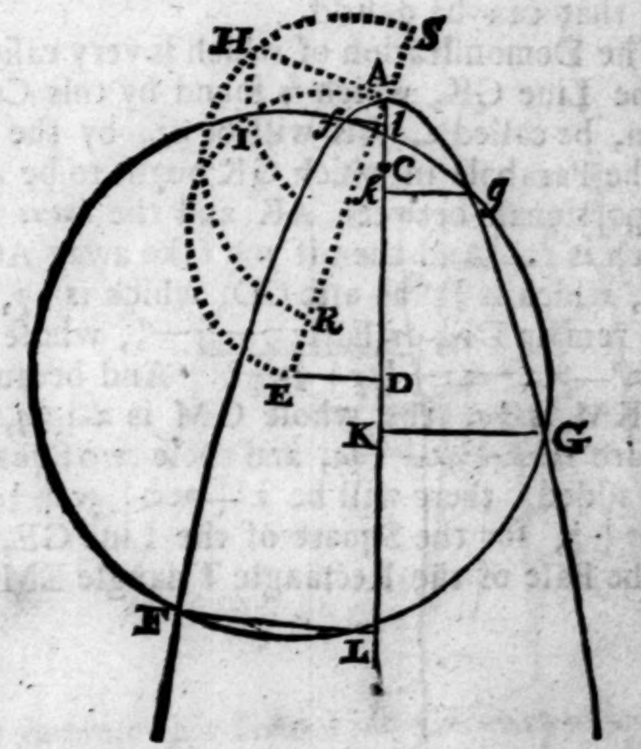


FG from the center E, whoſe Semidiameter is AE, if the Equation be only Cubical, that is, if we have not the Quantity r. But if the Quantity r be there, and affected with the Sign $+$, we are farther in this Line AE, produc'd on both ſides, to take on one ſide $AR=r$, and on the other ſide $AS=$ the *latus rectum* of the Parabola, which is $=1$, and deſcribing a Circle, whoſe Diameter



is RS, to erect AH perpendicular to AE, which will meet the Circle RHS in the Point H; which is the very Point through which the other Circle FHG ought to paſs. But if it be $-p$, we are moreover in another Circle, whoſe Diameter is AE, to inſcribe AI=AH found; and ſo the Point I will be found, through which the firſt Circle ſought FIG ought to paſs.

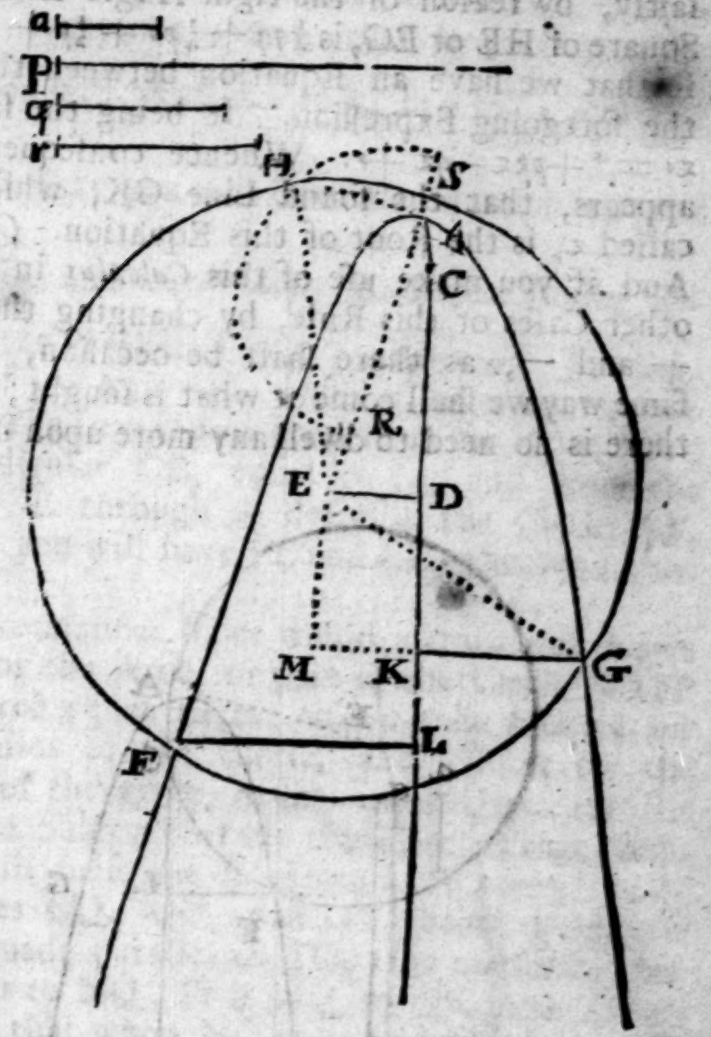
Where we muſt know that this Circle FG may cut or touch the *Parabola* in 1, 2, 3, or 4 Points, from which if Perpendiculars be let fall to the Axis, we ſhall have all the Roots of the Equation as well true as falſe. That is, if the Quantity *q* be affected with the Sign +, the true Roots will be thoſe perpendiculars which are found on the ſameſide of the *Parabola*, as is E the Centre of the



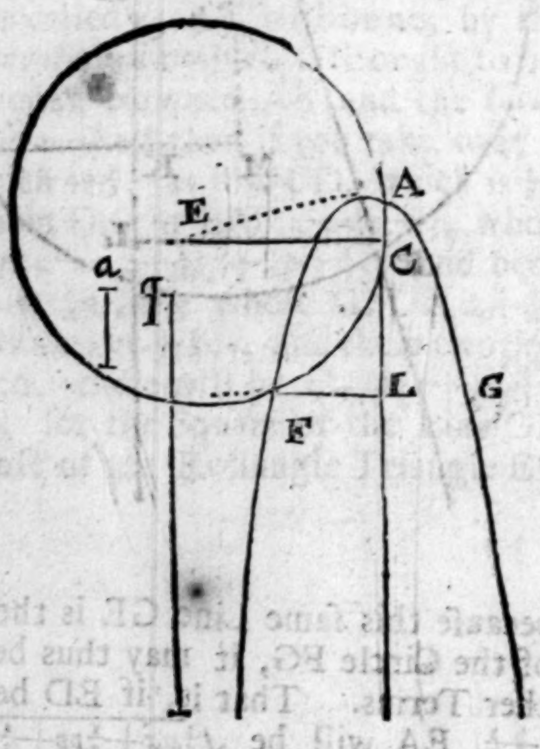
Circle, as (*ex. gr.*) FL, and the others as GK, will be the falſe Roots. But on the contrary, if this Quantity q hath the Sign —, thoſe will be the true Roots which are on the other ſide, and the falſe, or thoſe which are leſs than nought, will be found on that ſide where the Centre of the Circle E is. And laſtly, If this Circle neither cuts nor touches the *Parabola* in any Point, it is a Proof that the Equation admits of no Root either true or falſe, but only imaginary ones. So that this Rule is the moſt general and perfect of any that can be deſir'd.

The Demonſtration of which is very eaſie. For if the Line GK, which is found by this Conſtruction, be called z, AK will be zz, by the nature of the Parabola in which GK ought to be a mean Proportional between AK and the *latus rectum*, which is 1. And then if you take away AC from AK, which is $\frac{1}{2}$, as alſo CD, which is $\frac{1}{2}p$, there will remain DK, or EM $zz - \frac{1}{2}p - \frac{1}{2}$, whoſe Square is $z^4 - pzz - zz + \frac{1}{4}pp + \frac{1}{2}p + \frac{1}{4}$. And becauſe DE, or KM is $\frac{1}{2}q$, the whole GM is $z + \frac{1}{2}q$, whoſe Square is $zz + qz + \frac{1}{4}qq$, and theſe two Squares being added, there will be $z^4 - pzz + qz + \frac{1}{4}qq + \frac{1}{4}pp + \frac{1}{2}p + \frac{1}{4}$, for the Square of the Line GE, which is the Baſe of the Rectangle Triangle EMG.

But



But becauſe this ſame Line GE is the Semidiameter of the Circle FG, it may thus be explained in other Terms. That is, if ED be $\frac{1}{2}q$, and AD$\frac{1}{2}p$, $+\frac{1}{2}$, EA will be $\sqrt{\frac{1}{4}qq+\frac{1}{4}pp+\frac{1}{2}p+\frac{1}{4}}$, by reaſon of the right Angle ADE. And then ſince AH is a mean Proportional between AS, which is 1, and AR, which is r, it will be $\sqrt{r}$. And laſtly, by reaſon of the right Angle EAH, the Square of HE or EG, is $\frac{1}{4}qq + \frac{1}{4}pp + \frac{1}{2}p + \frac{1}{4} + r$: ſo that we have an Equation between this and the foregoing Expreſſion. It being the ſame as $z^4 = * + pzz - qz + r$. Whence conſequently it appears, that the found Line GK, which was called z, is the Root of this Equation. *Q. E. D.* And if you make uſe of this *Calculus* in all the other Caſes of this Rule, by changing the Signs + and −, as there ſhall be occaſion, by the ſame way we ſhall come at what is ſought; ſo that there is no need to dwell any more upon it.

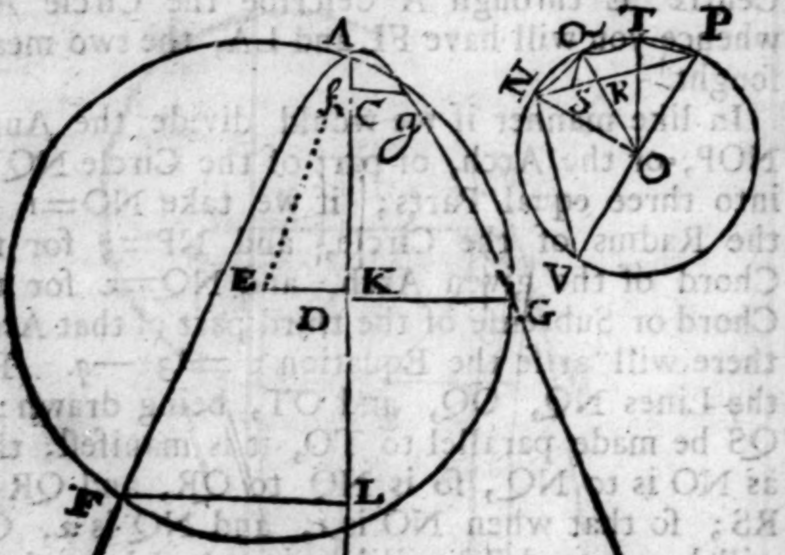


If therefore according to this Rule you would find two mean Proportionals between the Lines a and q, every one knows, that by puttiug z for one, you will have as a to z, ſo z to $\frac{zz}{a}$, and $\frac{zz}{a}$ to $\frac{z^3}{aa}$ ſo that you will have an Equation between q and $\frac{z^3}{aa}$, and $z^3 = {*}{*}aaq$. And then having deſcrib'd the Parabola FAG, together with the Segment of its Axis AC, which is $\frac{1}{2}a$, that is, one half of the *latus rectum* erect from the Point C the Perpendicular CE, equal to $\frac{1}{2}q$, and from the Centre E through A deſcribe the Circle AF, whence you will have FL and LA, the two means ſought.

In like manner if we would divide the Angle NOP, or the Arch, or part of the Circle NQTP into three equal Parts; if we take NO$=1$ for the Radius of the Circle, and NP$=q$ for the Chord of the given Arch, and NQ$=z$ for the Chord or Subtenſe of the third part of that Arch, there will ariſe the Equation $z^3 = {*}3z - q$. For the Lines NQ, OQ, and OT, being drawn; if QS be made parallel to TO, it is manifeſt, that as NO is to NQ, ſo is NQ to QR, and QR to RS; ſo that when NO is 1, and NQ is z, QR will be zz, and RS will be z^3. And becauſe RS or z^3 are the only hindrance why the Line NP, which is q, is not treble the Line NQ, which is z, you will have

$$q = 3z - z^3, \text{ or } z^3 = {*}\,3z - q.$$

And then having deſcrib'd the Parabola FAG, in which CA is equal to half of the principal *latus rectum*, if we take $CD=\frac{1}{2}$, and the perpendicular $DE=\frac{1}{2}q$, the Circle FA*g*G deſcrib'd in the Centre E through A will cut this Parabola in three Points F, *g*, and G, not reckoning the Point A, which is its *Vertex*, which ſhews, that in this Equation there are three Roots, that is, the two GK and *gk*, which are true ones, and a third, *viz.* FL, which is a falſe one; and of theſe two true Roots, the leſſer *gk* is that which we are to take for NQ the Line ſought. For the other GK is equal to NV, *viz.* to the Chord



of the third part of the Arch NVP, which with the remaining Arch NQP makes up the whole Circle. But the falſe Root FL is equal to thoſe two QN and NV taken together, as may be eaſily perceiv'd by the *calculus*.

It wouid be ſuperfluous for me to inſiſt in this place on any other Examples for the publick Benefit, ſince all Problems, which are only *Solid*, may be ſo reduced, as to have no other occaſion for this Rule for the Conſtruction of 'em, than as far as it ſerves to find out the two mean Proportionals, or to divide an Angle into three equal parts. Which will be known, if we conſider, that the difficulty of them is always comprehended under Equations, which riſe not beyond a *Biquadrate*; and that all thoſe which do ariſe to a *Biquadrate* may be reduc'd to a Square, by the help of ſome others, which ariſe only to a Cube; and laſtly, that their ſecond Term may be taken away. So that there is none of them which we cannot reduce to any of theſe three Forms.

$$z^3_3 = * - pz + q.$$
$$z = * + pz + p.$$
$$z^3 = * + pz - q.$$

But if you have $z^3 = * - pz + q$, the Rule (the Invention of which *Cardan* aſcribes to one *Scipio Ferreus*) teaches us that the Root is

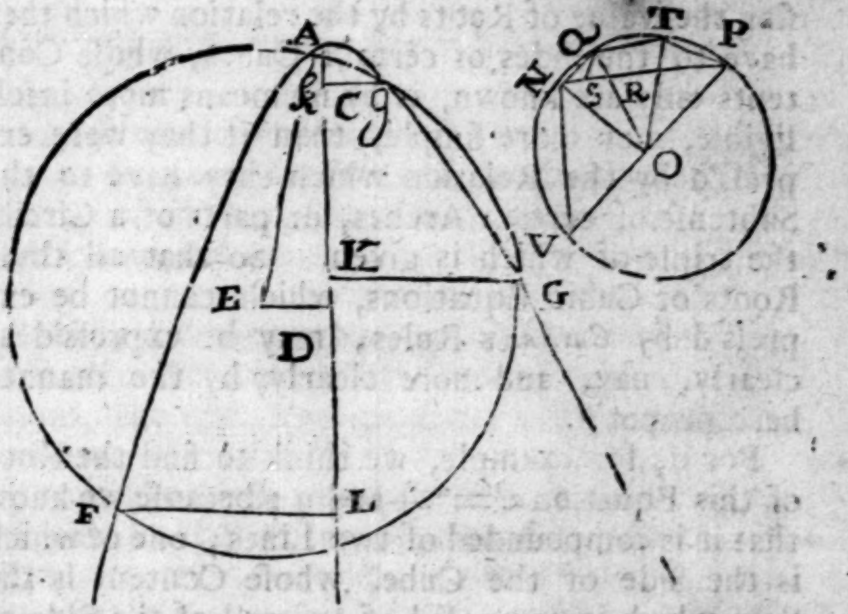
$$z = \sqrt{C.+\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}} - {}_3\sqrt{C.-\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}}$$

But as alſo if you have $z^3 = * + pz + q$, and the Square of half the laſt Term be greater than the Cube of the third part of the known Quantity of the laſt but one: the like Rule in a manner teaches us that the Root is

$$z=\sqrt{C.+\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^{3}}}+\sqrt{C.+\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^{3}}}$$

Whence it appears, that all Problems, the difficulty of which is reduc'd to an Equation of one of theſe two Forms, may always be conſtructed, ſo as to have no occaſion to make uſe of Conic Sections, unleſs it be to extract the Cubical Roots from certain given Quantities, that is, to find out two mean Proportionals between theſe Quantities and Unity.

And then if you have $z^3 = * + pz + q$, and the Square of half the laſt Term be not greater than the Cube of the third part of the known Quantity of the laſt Term but one; by ſuppoſing a Circle NQPV, whoſe Semidiameter NO is $\sqrt{\frac{1}{3}p}$, that is, a mean Proportional between the third part of the given Quantity p and Unity; and then by ſuppoſing the Line NP to be inſcrib'd in this Circle, which is $\frac{3q}{p}$, that is, which is to the other given Quantity q, as Unity is to the third part of p; only each Arch NQP, NVP is to be divided into three equal parts, and NQ, the Subtenſe of the third part of one of the Arches, together with NV the Subtenſe of the third part of the other Arch, will be equal to the Root ſought.



Laſtly, if there be $z^3 = * + pz - q$, by ſuppoſing again the Circle NQPV, whoſe Radius NO is $\sqrt{\frac{1}{3}p}$, and in which NP being written is $\frac{3q}{p}$, it will be NQ ſubtending the third part of the Arch NQP, together with the Roots ſought; and NV ſubtending the third part of the Arch NVP, the other Root. At leaſt if the Square of half the laſt Term exceed not the Cube of a third part of the known Quantity of the laſt Term but one. For if it ſhould be greater, the Line NP could not be inſcrib'd in this Circle, becauſe it would be greater than its Diameter. Which very thing would ſhew that the two Roots of this Equation are merely imaginary, and that there is no real, one, but a falſe one, which, according to *Cardan*'s Rule would be

$$\sqrt{C.\ \tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}} + \sqrt{C.\ \tfrac{1}{2}q - \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}}.$$

But we muſt obſerve, that this way of expreſſing the value of Roots by the relation which they have to the Sides of certain Cubes, whoſe Contents only are known, is by no means more intelligible, nor more ſimple, than if they were expreſs'd by the Relation which they have to the Subtenſe of certain Arches, or parts of a Circle, the triple of which is given. So that all thoſe Roots of Cubic Equations, which cannot be expreſs'd by *Cardan's* Rules, may be expreſs'd as clearly, nay, and more clearly, by the manner here propos'd.

For if, for example, we think to find the Root of this Equation $z^3 = * + pz + q$; becauſe we know that it is compounded of two Lines; one of which is the Side of the Cube, whoſe Content is the Sum which is compoſed of $\frac{1}{2}q$, and of the Side of the Square, whoſe Content is $\frac{1}{4}qq - \frac{1}{27}p^3$; and the Side of the other Cube, whoſe Content is the difference which is between $\frac{1}{2}q$, and the ſide of the Square, whoſe Content is $\frac{1}{4}q - \frac{1}{27}p^3$, (all which is the ſame that we learn from *Cardan's* Rule). There is no doubt but the Root of this $z^3 = * + pz - q$ may be known as diſtinctly, nay, and more diſtinctly, if we conſider it as inſcribed in a Circle, whoſe Semi-diameter is $\sqrt{\frac{1}{3}p}$, in which it ſtands for the Subtenſe of an Arch, the Subtenſe of whoſe triple is $\frac{3q}{p}$. Moreover, theſe Terms are nothing nigh ſo intricate as the former, and which will be rendred much more ſhort if we make uſe of ſome peculiar Marks to expreſs theſe Subtenſes, as is done by the Mark $\sqrt{}$C to expreſs a Cubical Root.

And ſo much concerning the Method of *Cartes*, as it is ſet down in the Authors own Words by Mr. *Paget*, the late Learned Maſter of the *Royal Mathematical School.*

But *this Rule for conſtructing ſolid Problems*, might come into the Author's Head from two Conſiderations, the one *Algebraical*, the other taken from *Geometry*.

For if the *Circle cuts a Parabola* in four Points, and you let fall Perpendiculars from the Interſections to the *Axis*: the Sum of them on one ſide of the *Axis* will be equal to the Sum of the Remainder, which are on the other ſide of the *Axis*. And if by chance *the Circle touch the Parabola*, then the double ordinate from *the point of Contact*, will be equal to the other two *taken together*. And the *famous Geometrician De la Hire*, in his 5*th* Book of *Conic Sections* demonſtrates, that this Property belongs not only to the *Parabola*, but alſo to *Ellipſes and Hyperbolas*, which are interſected by a Circle or other Sections.

Now having well-conſidered this Property of the *Parabola*, it followed from thence, that if the *Negative* and *Affirmative* Roots of any Equation ſhould deſtroy one another, or if the Sum of the Negative-Roots were equal to the Sum of the Affirmative (that is, *if the ſecond Term of the Equation were taken away*) then the Roots of that Equation might be accurately explained by ſuch *Perpendiculars*, let down from the Interſections to the *Axis* of the *Parabola*: ſo that nothing more would remain to finiſh *the Rule for Conſtruction*, but to diſcover the *Poſition and Magnitude* of a Circle which ſhould ſo cut the *Parabola*, that the Ordinates drawn from the Interſections, ſhould expreſs the Roots of the given Equation.

But this Circle would be known both by *Poſition and Magnitude*, if its Center, and alſo the Magnitude of the Radius were given by Poſition; that is, if it could be found, by what *Radius*, and at what *diſtance* from the Axis of the Parabola we ought to deſcribe it.

To this end we are to conſider the Lines, AD, DE, EH, EI, as wholly unknown, and the Values of them are to be found in the known Terms of the given Equation.

Suppoſe $AD = x$, $DE = y$, EH or $EI = v$, $KG = z$, (as before),

then you'll have $AK = \frac{zz}{a}$, $DK = EM = \frac{zz}{a} - x$, $MG = z + y$, and by conſequence $EG^q \ (= EM^q + MG^q) = \frac{z^4}{aa} - \frac{2xzz}{a} + xx + yy + 2yz + zz \ (= EH^q) = vv$, by reducing and ordering the Equation $z^4 =$.

$$\begin{array}{l} +2ax \\ -aa \end{array} zz - 2aayz \begin{array}{l} +aavv \\ -aaxx \\ -aayy \end{array}$$

Compare this with any given Equation (*ver. gr.*

$z^4 = + apzz - aaqz + a^3r$, and by comparing the *homologous* Terms the following Equalities will ariſe, *viz.* 1. $2ax - aa = + ap$, $\therefore x = \frac{1}{2}a + \frac{1}{2}p$,

2. $2aay = aaq$, $\therefore y = \frac{1}{2}q$,

3. $aavv - aaxx - aayy = a^3r$, $\therefore vv = ar + xx + yy$, and ſubſtituting the found Values of x and y, you will have $vv = ar + \frac{1}{4}aa + \frac{1}{2}ap + \frac{1}{4}pp + \frac{1}{4}qq$.

Now hence it is plain, that inſtead of the Line AD we are to take in the Axis of the Parabola, $\frac{1}{2}a + \frac{1}{2}p$, or half the *Latus Rectum*, together with half of the known Quantity of the third Term of the Equation, as being in this place adfected with a poſitive Sign. Beſides, the length of DE (which determines the Center) is to be taken equal to half of the known Quantity of the fourth Term.

Laſtly, the Radius EH or v ought to be ſuch, that the Square of it may be equal to the Square of the Line AE $(= \frac{1}{4}aa + \frac{1}{2}ap + \frac{1}{4}pp + \frac{1}{4}qq)$ together with a certain Square which ſhall be equal to the Rectangle ar; that is to ſay, in the produced Line AE we ought to take on one ſide, from the point A, the length of r, and on the other ſide the length of the *Latus Rectum* a or 1, and making the Line SR compounded of both, the Diameter of the Semicircle, from the foreſaid Point A we are to erect a perpendicular AH to the Semicircle, which Line AH will therefore be a mean Proportional between the two Segments of the Diameter, and therefore its Square will be equal to the Rectangle ar, whence the Line EH gives the Magnitude of the Radius ſought.

But if the Quantity r had been adfected with a negative Sign, then we had found $vv = xx + yy - ar$, and therefore the Square of the Radius ſought equal to the *difference* of the Squares of the Lines AE and AH.

From whence we learn, that the former Conſtruction ſtanding, and a Semicircle being deſcribed on the Diameter AE, we ought to ſet off upon it the Line AI equal to AH, and then to deſcribe a Circle from the Center E (as before) with the Radius EI.

If the Quantity r be imagin'd equal to nought, or to be altogether wanting, then the Equation will be turn'd into a *Cubic* one, and conſequently the conſtruction of it is very eaſily deduc'd.

Becauſe $a^3r=0$, $\therefore$ $vv=xx+yy$, $=AE^q$, and by conſequence the Circle ought to paſs through the Vertex A of the Parabola.

And theſe are the things which in *Cartes's* Method, are requir'd to the Conſtruction of theſe Problems.

Now hence follows another Method of Conſtructing Problems of this kind, in which the Perpendiculars let fall from the Interſections of the Circle and the Parabola, are not related to the *Axis it ſelf*, but to ſome *Diameter* (which is Parallel to the Axis from the nature of a Parabola)

For ſuppoſe the Equation $x^4+px^3-qx^2-rx+s=0$, to Conſtruct which in *Cartes's*, manner, take away the *ſecond* Term, that is to ſay, by ſuppoſing $z=x+\frac{1}{4}p$, $\therefore$ $x=z-\frac{1}{4}p$, which being every where ſubſtituted for x, there ariſes

$$\begin{array}{r}
z^4-z^3p+\frac{3}{8}z^2p^2-\frac{1}{16}zp^3+\frac{1}{256}p^4=x^4\\
+z^3p-\frac{3}{4}z^2p^2+\frac{3}{16}zp^3-\frac{1}{16}p^4=px^3\\
-qz^2+\frac{1}{2}qpz+\frac{1}{16}qp^2=-qx^2\\
-rz+\frac{1}{4}rp=-rx\\
+s\qquad\qquad
\end{array}$$

That is, taking away the contradictory Terms,

$$z^{4}\begin{array}{l}-\frac{3}{8}p^{2}\\-q\end{array}zz\begin{array}{l}+\frac{1}{8}p3\\+\frac{1}{2}qp\\-r\end{array}z\begin{array}{l}-\frac{3}{256}p^{4}\\-\frac{1}{16}qp^{2}\\+\frac{1}{4}rp\\+s\end{array}=0$$

Therefore *Cartes's Vertical* Line AD conſiſts of AC or $\frac{1}{2}$, and the Quantity $-\frac{1}{16}p^{2}-\frac{q}{2}$ which is to be plac'd downwards from the Point C; and the Perpendicular DE$=\frac{1}{16}p^{3}+\frac{1}{4}qp-\frac{1}{2}r$ to be taken on the right Hand from the Point D ſuppoſing $r>\frac{1}{8}p^{3}+qp$.

If a Line be drawn which touches the Parabola in the *Vertex* A, and on it be taken the diſtance $\frac{1}{4}p$ to the right ſide of the *Axis*, and thro' the end of it be drawn a Line parallel to the *Axis*, the exterior part of the Diameter, *viz.* that intercepted between the Curve and its vertical Tangent, will $=\frac{1}{16}p^{2}$ (as being the *abſciſs* anſwering to the ordinate $\frac{1}{4}p$). Wherefore the Segment of the Diameter which lies between the Curve and the Line DE *produc'd*, will be equal to AD$-\frac{1}{16}p^{2}$ $=\frac{1}{2}+\frac{3}{16}p^{2}+\frac{q}{2}-\frac{1}{16}p=\frac{1}{2}+\frac{q}{2}+\frac{1}{8}p^{2}$: which is the Rule of the famous Mr. *Baker*, for determining his vertical Line, for this Form of the Equation.

Laſtly, Becauſe DE$=\frac{1}{16}p^{3}+\frac{1}{4}qp-\frac{r}{2}$, therefore the diſtance between the Point E, and the Diameter aforeſaid is neceſſarily equal to $\frac{1}{16}p^{3}+\frac{1}{4}qp-\frac{r}{2}+\frac{1}{4}p$; which likewiſe is the forementioned Author's *Formula* for finding out his *tranſverſe* Line, or that which is Perpendicular to the Diameter.

And from theſe two his *Central Rule* is form'd, or his Rule for finding the place of the Center, whence a Circle being deſcrib'd, paſſing through the Vertex of the *Diameter* in *Cubic* Equations, but in *Biquadratics* through the Point diſcover'd the ſame way as before, will cut the Parabola in as many Points, as the Equation hath real Roots.

To this we may add the following *Inveſtigation* of the ſaid Rule of Mr. *Baker*, which proceeds much after the ſame manner with that which went before of *Des Cartes*: Only the Reader is deſir'd to ſupply the Figure proper on this occaſion; which is ſo very little different from the others before taken from *Des Cartes*, that the drawing of it can be no trouble at all to thoſe who have before ſeen the way of theſe Conſtructions; and as for others who have not, it may eaſily be ſupply'd for them.

However, to make all as eaſie as poſſible, without drawing a Figure. Suppoſe the Equation to be this, *viz.* $z^4 + pz^3 - qz^2 - rz + s = 0$, and let the Vertex of the Parabola be imagin'd to be *a*, the Axis *a*Y, the vertical Tangent *a*E, the *Diameter* us'd for the Conſtruction EADO, which cuts the Curve in A (the Vertex belonging to that Diameter) the Lines determining the Centre of the Circle ſought, AD and DH, of which AD is a Segment of the Diameter, and DH at right Angles thereto, ſo that H is the Centre. Let the two *Radii* of the Circle (whoſe Values are to be equall'd) be repreſented by HN and HL, the Point N (in the former of which two *Radii*) is common to the Circle and the Parabola, ſo that NO let fall perpendicular from that Point N to the Diameter EADO, may expreſs the Root z. Laſtly, Let NO be conceiv'd to be produc'd till it cuts the Axis in Y, and a Line from the Centre H to be let fall upon the ordinate NY produc'd, which Perpendicular (ſuppoſe) meets it in the Point B. So that from what has been ſuppos'd, we have theſe Lines equal one to the other, *viz.* $aE = YO$, $EO = aY$, $HB = DO$. Put $aE = b$, $AD = x$, $DH = y$, $HL = v$, $ON = z$, Parameter $= a = 1$, $BN = y + z$.

Therefore $YN = z + b$, $EA = \frac{bb}{a}$, $aY = EO = \frac{zz + 2zb + bb}{a}$, $AO = \frac{zz + 2zb}{a}$, $DO = \frac{zz + 2zb - ax}{a} = HB$.

Becauſe HN and HL are *Radii* of the ſame Circle, therefore this Equation,

$$z^4 + 4bz^3 \begin{array}{l} +4bb \\ -2ax \\ +aa \end{array} zz \begin{array}{l} -4bax \\ +2aay \end{array} z \begin{array}{l} +aaxx \\ +aayy \\ -aavv \end{array} = 0$$

Compare this with the Equation propos'd, then

1. $4b = p$, or $b = \frac{1}{4}p$, which is the diſtance of the Diameter ſought from the Axis, taken in the vertical Tangent.

2. $4b^2 - 2ax + aa = -q$, and ſubſtituting the Value of b already found, we have $x = \frac{1}{8}p^2 + \frac{q}{2} + \frac{1}{2}$.

3.

3. $2aay - 4bax = -r$, or $y = \frac{4bax - r}{2aa}$, and ſubſtituting the Values of x and b found before, we have $y = \frac{1}{16}p^3 + \frac{1}{4}p + \frac{1}{4}pq - \frac{r}{2}$.

4. $aaxx + aayy - aavv = +s$, or $vv = xx + yy - s$, or (ſince $a =$ unity) $vv = xx + yy - as$: From whence *Cartes*'s Rule alſo for the Radius of the Circle is evident.

And thus may Mr. *Baker*'s Central Rule for any poſſible Equation be *inveſtigated*.

The Reader is deſired to refer what follows, to the Places taken notice of before in the Appendix.

At Page 40. Line 13. after the words *to find*, proceed thus.

Or from the Equation which reſults, you may, after another manner, find out the Expreſſion of the Quantity y in the Terms of v, but by a *Series* (tho' only one, yet) much more compounded and perplex'd.

At Pag. 42. Line the laſt, after theſe Words, *whence a way is open'd*, add what follows; But two ways alſo of expreſſing the Quantity y by v, and the *known* Quantities may be thus otherwiſe obtain'd.

Becauſe $\overline{1+y}^{n} = \overline{1+v}^{\frac{1}{s}}$, $\therefore 1+y = \overline{1+v}^{\frac{1}{sn}}$, whence you have y by a *Series* of the Powers of v.

Again, $\overline{1+y}^{\frac{1}{n}} = \overline{1+v}^{r}$, $\therefore 1+y = \overline{1+v}^{rm}$, and thence is found another *Series* for y. Which two

Series being made equal to one another, you have an Equation for v.

But I had rather make uſe of *one General Rule in all Caſes*, and therefore always proceed by a *Method for extracting the Root from an infinite Equation*, tho' in the Caſe brought it may be performed without it, and by means of only a *Binomial Theorem*.

But we will joyn another Example to what went before, by which the uſe of theſe Methods will more clearly appear, in ſeparating and expreſſing unknown Quantities.

Let the Equation be
$$\begin{cases} az+bz^2+cz^3,\&c.=\overline{dy+ey^2+fy^3}^{\,n},\&c. \\ gy+hy^2+iy^3,\&c.=\overline{kx+lx^2+mx^3}^{\,r},\&c. \\ nx+ox^2+px^3,\&c.=\overline{qv+rv^2+ſv^3}^{\,m},\&c. \\ lv+nv^2+qv^3, \quad =\overline{cz+dz^2+bz^3}^{\,s},\&c. \end{cases}$$

For in the firſt Equation, by help of the *Infinitonomial Theorem* (which is alſo owing to the afore-cited Mr. *de Moivre*) the Infinite Series $dy+ey^2+fy^3$, *&c.* is elevated to the Power n.

And thence by the other Theorem (for extracting a Root from an Infinite Equation) you have z in an Infinite Series of the Powers of y.

Again in the very ſame manner, by the *fourth* Equation $lv+nv^2+qv^3$, &c. $=\overline{cz+dz^2+fz^3,}^{\,s}$&c. z is expreſs'd in the Terms of v.

Therefore ſince you have z expreſs'd by two Infinite Series, *viz.* of the Powers y and v, you'll alſo have y in the Terms of v.

In like manner by the 2*d* Eq. $gy+hy^2+iy^3$,&c. $= \overline{kx+lx^2+mx^3, \&c.}^{\,r}$, x is expreſs'd in y, and by the 3*d*. Equat. $nx+ox^2+px^3$,&c. $=\overline{qv+rv^2+ſv^3,\&c.}^{\,m}$

you have alſo x in v. Wherefore by the means of two Infinite Series, by which x is expreſs'd both in y and alſo in v, another Equation will be found, by which y will be a *ſecond time* expreſs'd in the Terms of v. And from theſe two *Series*, which exhibit ſo many Values of the Quantity y, there ariſes a determinate Equation, by which the Quantity v is fully expreſs'd.

FINIS.
